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ABSTRACT

This volume contains three teacher-developed u ts for eleventh grade mathematics students. It serves as an intro ction to logarithms, matrices, and functions. Units include statements of objectives, content discussions, worksheets, and exercises. In the logarithm unit the emphasis is on calculation, while in the matrices and functions units development and proof are considered as well. Related volumes in the series are SE 016 615, SE 016 617, and SE 016 618. (LS)

THIRD YEAR MATH

MADEHEIM

SAHUARITA HIGH SCHOOL

CAREER

CURRICULUM

PROJECT

COURSE TITLE: THIRD YEAR MATH

PACKAGE TITLE: LOGARITHMS

BY

JAMES MADEHEIM

ADVANCED MATH

OBJECTIVES

1. Given a series of numbers, you will be able to multiply and divide them using logarithms with 80% accuracy.

2. Given a 2 x 2 mat4ix, you will be able to:

    1. transpose it

    2. find its determinant

    3. find its inverse, if it has one

    4. multiply it times another 2 x 2 matrix.

Given a test covering the above material, you will be able to complete 70% of it accurately.

Rationale:

Scientists and engineers at times find themselves presented with a problem such as the one below.

$$x = \sqrt[5]{67} \qquad \text{Find } x.$$

The easiest way to do this problem without a sophisticated calculator, is by using logarithms.

This package offers a review of working with fractional exponents and shows you how you can invent logarithms, and then use them.

Behavioral objective:

Given a series of numbe 3, you will be able to multiply and divide them using logarithms with 80% accuracy.

Pre-test:

Simplify by using logs. (no calculator or computer)

$$\frac{(.921)^7(762)(1.57)}{(\sqrt[5]{820})(\sqrt[7]{1500})}$$

Information Sources:

Textbook - Modern Algebra and Trig. Dolciani

Read Data Brief # 1 "Exponent Arithmetic"

Read Data Brief # 2 "Linear graph Paper"

Read Data Brief # 3 "Semi-log Graph Paper"

Read Data Brief # 4 "Inventing Logs"

Read Data Brief # 5 "Computations with Logs"

In your text, do written exercises p. 354, #1-16, all

Data Brief # 1

Operating with exponents.

$2 \times 2 \times 2 \times 2 = 2^4$

$3 \times 3 = 3^2$

$5 = 5^1$

The exponent tells how many times the base is used as a factor.

$a(a)(a)(a)(a)(a) = a^6$

$b \times b \times b \times b \ . \ . \ . \times b = b^n$ when there are n factors.

c^2 is read as "c squared, or c to the second power"

d^3 is read as "d cubed, or d to the third power"

a^4 is read as "a to the fourth"

b^9 is read as "b to the ninth"

Consider!! $2^3\ 2^2 = (2 \cdot 2 \cdot 2)(2 \cdot 2) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5$

$3^4\ 3^7 = (3 \cdot 3 \cdot 3 \cdot 3)(3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

$= 3^{11}$

In General $a^m \cdot a^n = a^{m+n}$

Consider!! $\frac{2^5}{2^3} = \frac{2\cdot2\cdot2\cdot2\cdot2}{2\cdot2\cdot2}$ $\frac{2}{2}\cdot\frac{2}{2}\cdot\frac{2}{2}\cdot2\cdot2 = 4 = 2^2$

In General $\frac{a^m}{a^n} = a^{m-n}$

$\frac{2}{2} = 1$ $\frac{789}{789} = 1$ $\frac{1056}{1056} = 1$ $\frac{a^m}{a^m} = a^{m-m} = a^0 = 1$

$2^0 = 1$ $789^0 = 1$ $1056^0 = 1$

Consider!! $\frac{a^2}{a^7} = a^{2-7} = a^{-5}$

But $\frac{a^2}{a^7} = \frac{a\cdot a}{a\cdot a\cdot a\cdot a\cdot a\cdot a\cdot a} = \frac{1}{a^5}$

In general $a^{-n} = \frac{1}{a^n}$

Consider!! $(5^3)^2 = (5^3)(5^3) = 5^6$

In General $(a^m)^n = a^{mn}$

Example: $\left(\frac{1}{2}\right)^{-2} = \frac{1}{\left(\frac{1}{2}\right)^2} = \frac{1}{\frac{1}{4}} = 4$ $\frac{1}{3^{-4}} = 3^4$

Consider!!! $a^{\frac{1}{2}} \cdot a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1$

$$a^{\frac{1}{2}} = a^{\frac{1}{2}}$$

$$9^{\frac{1}{2}} \cdot 9^{\frac{1}{2}} = 9^1$$

$$9^{\frac{1}{2}} = 9^{\frac{1}{2}}$$

This means some number times itself equals 9. Therefore -

$$9^{\frac{1}{2}} = 3$$

$$9^{\frac{1}{2}} = \sqrt{9}$$

In General

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

$$8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} \cdot 8^{\frac{1}{3}} = 8$$

$$8^{\frac{1}{3}} = 2 = \sqrt[3]{8}$$

Consider!!! $25^{\frac{1}{4}} = (5^2)^{\frac{1}{4}} = (5)^{\frac{1}{2}} = \sqrt{5}$

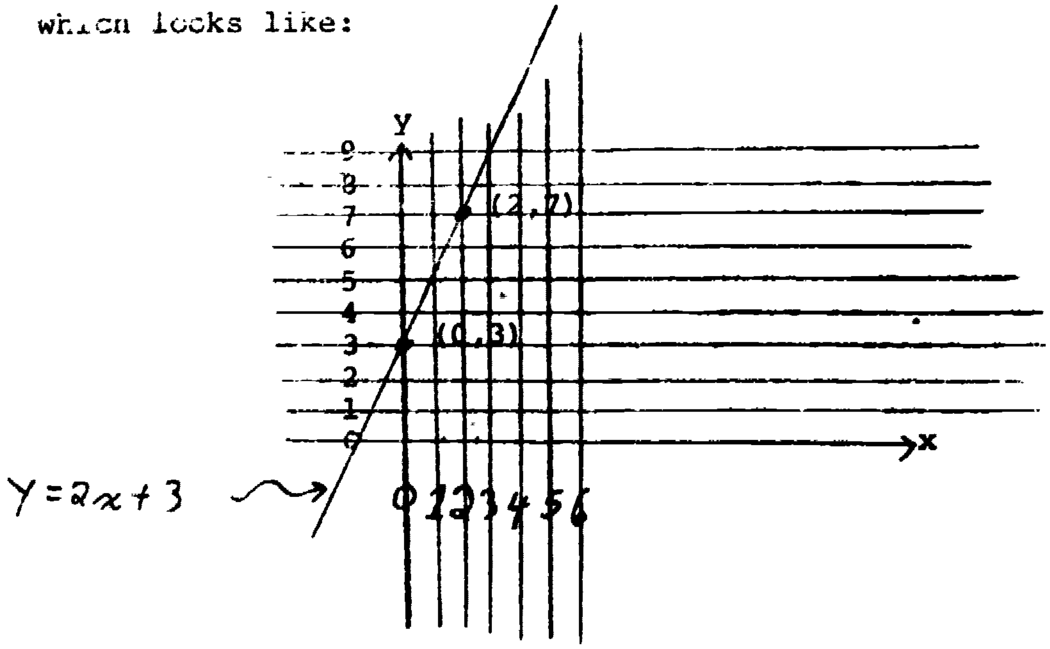
$$\sqrt[6]{8^3} = 8^{\frac{3}{6}} = 8^{\frac{1}{2}} = \sqrt{8} = 2\sqrt{2}$$

In General $a^{\frac{b}{c}} = \sqrt[c]{a^b} = (\sqrt[c]{a})^b$

$$10^{\frac{2}{3}} = 10^{.667}$$

Data Brief # 2

We review briefly what the graph of a function looks like. In graphing the function $y = 2x + 3$, we obtain a graph which looks like:



We observe that when x is the value 0, y is the value 3.

This is written as (0,3)

When $x = 2$, $y = 7$, and the point (2,7) is plotted. When $x = 1.25$, $y = 5.5$ ($y = 2(1.25) + 3$) This value for y is easier to calculate from the equation than it is to look at the graph, and try to calculate it.

But sometimes it is easier to "guess" at a value on a graph than it is to calculate it!!

Data Brief # 3

Suppose we wish to multiply 1000 and 100. The product is 100,000. But lets try to multiply by adding (exponents).

$1000 = 10^3$

$100 = 10^2$ $\quad 10^3 \times 10^2 = 10^{2\ \ 3} = 10^5$

MATHEMATICIANS ARE LAZY !!!!!!

Isn't it easier to add 2 + 3 instead of multiplying 1000 x 100.

How about 2000 x 100 $\quad 2 \times 10^3 \times 10^2 = 2 \times 10^5 = 200000$

$3000 \times 600{,}000{,}000 = 3 \times 10^3 \times 6 \times 10^8 = 18 \times 10^{11}$

$= 1800000000000$

Now: $\quad 20 = 2 \times 10$

$200 = 2 \times 10^2$

In the above two examples, you can't simplify any further, and you can't find out the actual value of 2×10^2 because you don't know how to express 2 as a power of 10.

Let $2 = 10^x$ then $2 \times 10^2 = 10^x\ 10^2 = 10^{x+2}$

2×2 would be 10^{2x}

Thus extending this system for all numbers besides 2, we could multiply by <u>adding</u> exponents!!!!!

We will now attempt to find the powers of 10 for such numbers as 2, 3, 5, 6, etc., so that we can use exponents to multiply any two numbers together.

For example:

$$2 \times 6 = 12$$

We will rewrite this problem as

$$2 \times 6 = 10^h \times 10^y = 10^{h+y}$$

where $10^h = 2$ and $10^y = 6$

Thus we can add the exponents h and y in stead of multiplying the numbers 2 and 6.

Keep in mind that this is the goal of this package: to add instead of multiply.

At this point you should ask the teacher for a demonstration on using semi-log graph paper.

Data Brief # 4

$$2 = 10^x$$

We will now try to approximate the above value of x by numerical methods.

$$10^0 = 1$$

$$10^1 = 10$$

There is a number x such that $10^x = 2$

$$10^0 = 1$$

$$10^x = 2$$

$$10^1 = 10$$

Thus x is greater than 0 and less than 1.

We now will try to find 2 as 10 to some power.

$2 = 4$ $\quad 2^2 = 4$

$2^3 = 8$

$2^4 = 16$

$2^5 = 32$

$2^6 = 64$

$2^7 = 128$

$2^8 = 256$

$2^9 = = 512$

$2^{10} = 1024$

Now $1024 = 1000(1.024) = 10^3(1.024)$

But $1.024 \approx 1$ (The waving equals sign means approximately.)

hence

$$2^{10} \approx 10^3$$

But

$$2^{10} = (2^1)^{10} \text{ and } 10^{.3} = (10^{.3})^{10}$$

Thus

$$(2^1)^{10} \approx (10^{.3})^{10}$$

Or

$$2 \approx 10^{.3}$$

2(2) will become $10^{.3}(10^{.3}) = 10^{.6}$

Thus <u>multiplication</u> becomes the <u>addition</u> of exponents!!

Data Brief # 5

$$\frac{10^{.2} \times 10^{.9} \times 10^{3.2}}{10^{2.4} \times 10^{.4}} = \frac{10^{4.3}}{10^{2.8}} = 10^{1.5} = 31.6$$

$$\frac{12 \times 5}{3} = \frac{10^{1.08} \times 10^{.7}}{10^{.48}} = \frac{10^{1.78}}{10^{.48}} = 10^{1.30} = 20$$

$$\frac{630 \times 7.91}{321} \approx \frac{10^{2.8} \times 10^{.9}}{10^{2.5}} = 10^{1.2} = 15.8$$

Data Brief # 6

The teacher will demonstrate the use of the log tables in the back of your textbook.

Activity # 1

Complete the following problems:

1. $5^3(5) =$

2. $x^3 \cdot x^4 =$

3. $b \cdot b \cdot b \cdot b \cdot b \cdot b =$

4. $3 \cdot 3 \cdot 3 \cdot 3 =$

5. $a \cdot a \cdot c \cdot c \cdot c \cdot x \cdot x \cdot h =$

6. $\frac{x^5}{x^2} =$

7. $\frac{3x^2}{9x^3} =$

8. $\frac{a^5}{a^5} = \quad (a \neq 0)$

9. $a^{-5} a^7 b^3 b^{-4} =$

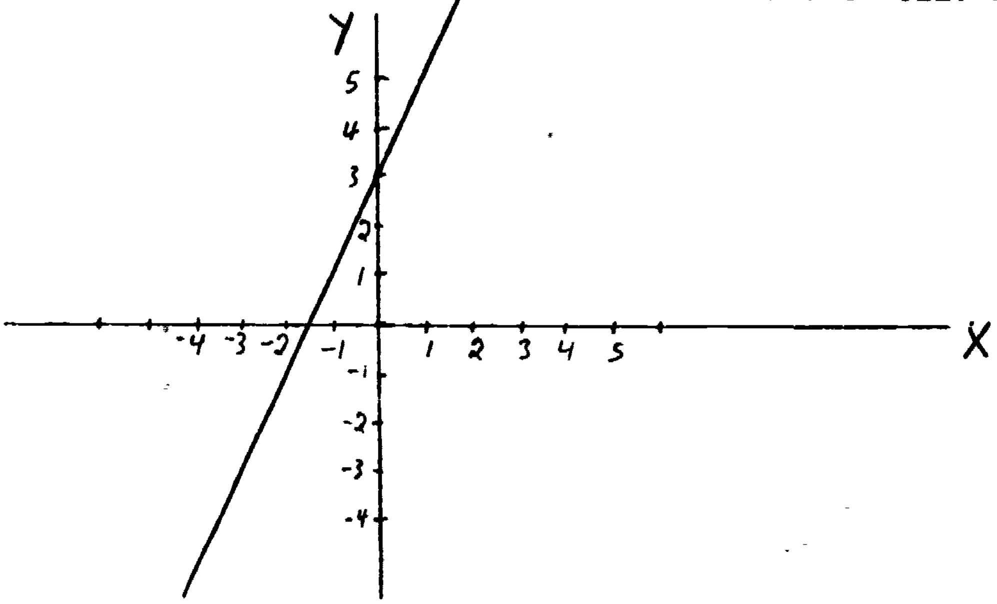
10. $\left(\frac{1}{3}\right)^{-2}$

In the textbook p. 335 1 - 39, odd problems and # 40

p. 337 1 - 16 all !!

Activity # 2

THIS IS A REVIEW OF WHAT THE GRAPH OF A FUNCTION LOOKS LIKE. IF YOU GRAPH $Y = 2X + 3$, YOU GET A GRAPH WHICH LOOKS LIKE THE FOLLOWING:



YOU CAN OBSERVE THAT WHEN X IS 0, Y IS 3, AND WHEN X IS 2, Y IS 7.

1. BY READING FROM THE GRAPH, WHAT VALUE DOES Y HAVE WHEN $X = 1$? ________

2. NOW CALCULATE THE VALUE OF Y USING THE FORMULA $Y = 2X + 3$ WHEN $X = 1$. ________

3. WHAT VALUE OF X CAUSES Y TO TAKE ON THE VALUE OF 0? ________

SUPPOSE WE NOW WISH TO GRAPH $Y = 2^X$. PICK A VALUE FOR X AS LISTED IN THE TABLE BELOW AND DETERMINE THE CORRESPONDING VALUE FOR Y.

VALUE OF X	VALUE OF Y
6	
5	
4	
3	
2	
1	
0	
-1	
-2	
-3	
-4	
-5	

PLOT THE POINTS YOU OBTAIN IN THIS TABLE ON A GRAPH. USE TWO DIFFERENT SHEETS OF GRAPH PAPER. ONE FOR THE VALUES OF X FROM 1 TO 6, AND THE OTHER FOR THE VALUES OF X FROM -5 TO 1.

NOW OBSERVE YOUR GRAPH AND ANSWER THE FOLLOWING QUESTIONS.

1. THE VALUE OF Y FOR $Y = 2^X$ IS ALWAYS POSITIVE TRUE F...

2. AS X INCREASES IN VALUE, Y ALSO INCREASES IN VALUE TRUE FA...

3. WITH A CAREFULLY DRAWN GRAPH, VALUES OF Y AND X XAN BE READ APPROXIMATELY TO 2 OR 3 DECIMAL PLACES TRUE F...

4. USING YOUR GRAPH, WHAT IS THE VALUE OF Y FOR X = 2.5 ________

5. USING YOUR GRAPH, WHAT IS THE VALUE OF Y FOR X = 1.5? ________

6. ALSO FIND THE VALUE OF Y FOR X = .5 ________

7. WHEN X = .3, WHAT IS THE VALUE FOR Y? ________

PROBLEMS 4 THRU 7 USE A PROCESS CALLED INTERPOLATION.

Activity # 3

Graph the function $y = 10^x$ using the table below after you have filled it in.

Value of x	Value of 10^x	Value of y
5		
4		
3	10^3	1000
2		
1		
0		
-1		
-2		
-3		

Graph this function using 3-cycle semi-log paper. Only use the values of x from -1 to 2.

Using your graph, you will obtain the following values;

$10^{.1} =$ $\qquad$ $10^{.4} =$ $\qquad$ $10^{.7} =$

$10^{.2} =$ $\qquad$ $10^{.5} =$ $\qquad$ $10^{.8} =$

$10^{.3} =$ $\qquad$ $10^{.6} =$ $\qquad$ $10^{.9} =$

Activity # 4

$6^9 = 10,077,696$

$6^9 \approx 10^7$

Remember $2 \times 3 = 6$ and you already know $2 = 10^{.3}$

$N = 10^x$

N	x
2	.3
3	
4	
5	
6	
7	
8	
9	
10	1

More hints: $2 \times 5 = 10$, $7 \times 7 \approx 50$

Adding exponents from the above table, how close do you get to the actual value for

$$5 \times 5 \times 4$$

Activity # 5

Graph $N = 10^x$ on 1 cycle semi-log paper, x = 0 to 1. Make each tenth equal to 5 divisions across. Then fill in the tables below

N	x
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	
16	
17	
18	
19	
20	
21	
22	
23	
24	
25	
26	
27	
28	
29	
30	

N	x
	.1
	.2
	.3
	.4
	.5
	.6
	.7
	.8
	.9
	1.0
	1.1
	1.2
	1.3
	1.4
	1.5
	1.6
	1.7
	1.8
	1.9
	2.0
	2.1
	2.2
	2.3
	2.4
	2.5
	2.6
	2.7
	2.8
	2.9
	3.0
	3.1
	3.2
	3.3
	3.4
	3.5
	3.6
	3.7
	3.8
	3.9
	4.0

In the :ollowing problems use the tables on the preceeding page. You may not use the calculator except to check your work. Use the powers of ten as in the data brief.

1. $$\frac{10^{.7} \times 10^{1.2} \times 10^{5.8}}{10^{2.4} \times 10^{1.1}} =$$

2. $$\frac{10^{2.6} \times 10^{2.7} \times 10^{3.9}}{10^{3.1} \times 10^{3.1}} =$$

3. $$\frac{39.8 \times 126 \times 25.1 \times 1.58}{501 \times 3.16} =$$

4. $$\frac{794 \times 15.8 \times 1.26 \times 31.6}{63.1 \times 251} =$$

Post-test:

Open book test. Use logs only, no calculator.

Find x

1. $x = \frac{756}{1.67}$

2. $x = \frac{989}{144}$

3. $x = \sqrt[5]{87}$

4. $x = \sqrt[11]{5670}$

5. $x = \sqrt[3]{7500}$

6. $x = \sqrt[19]{9070 \times 10^{16}}$

7. $X = \frac{857 \times 682 \times 4.01}{1.67 \times 947}$

Extra credit

Given $2 = 10^{.3}$ $3^{12} = 531{,}441$ $3 = 10^{x}$

Find x

Your method is what will determine your grade.

SAHUARITA HIGH SCHOOL

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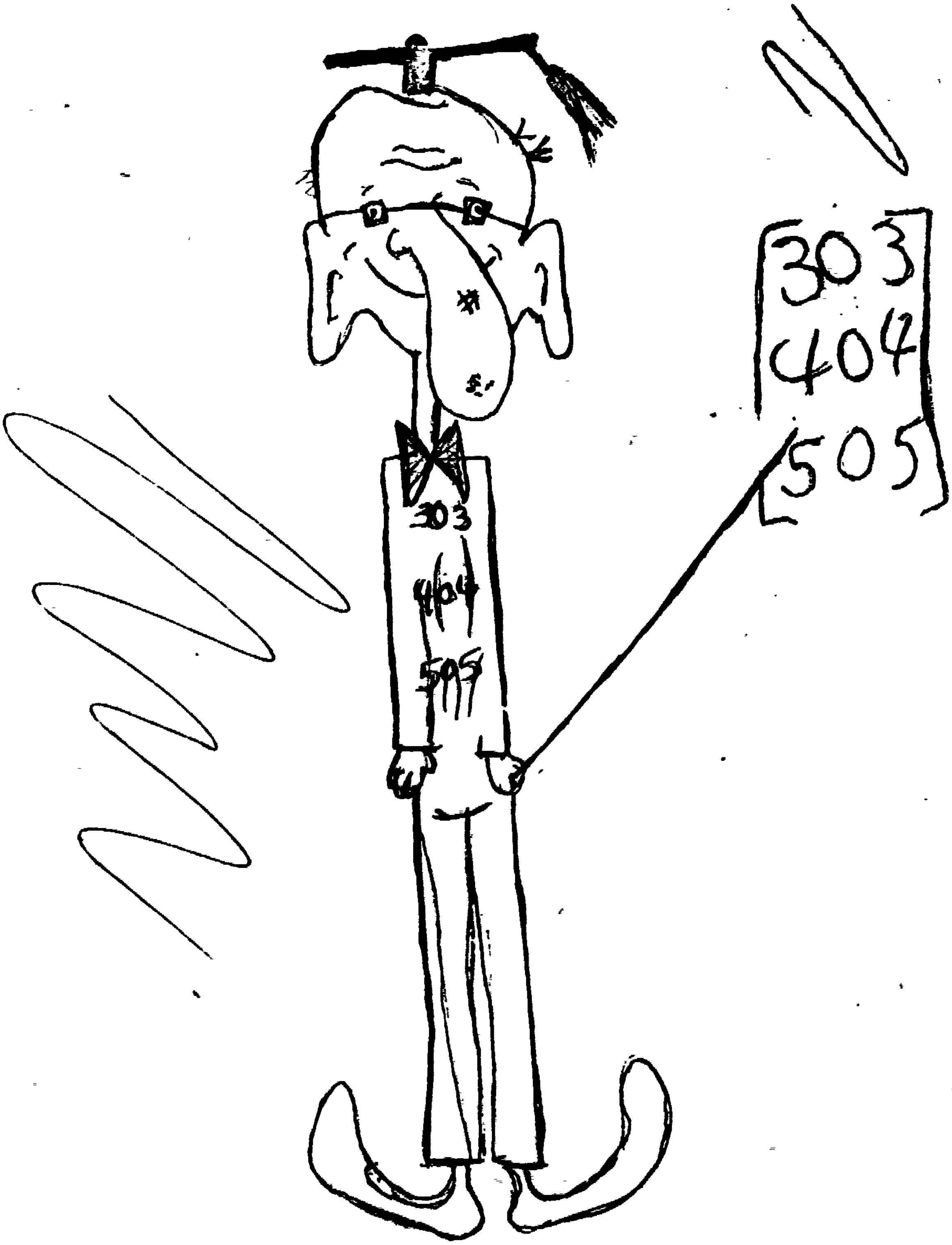
COURSE TITLE: THIRD YEAR MATH

PACKAGE TITLE: MATRICES

BY

JAMES [illegible]



PROFESSOR MATRIX

Rationale:

Engineering statics problems use determinants. Determinants are part of Matrix algebra.

This package develops some of the basic ideas of Matrix algebra. It is hoped that after completing this package, you will see the necessity for defining the laws of an algebraic system.

Behavioral objectives:

Given a 2 x 2 matrix, you will be able to -

1. transpose it
2. find its determinant
3. Find its inverse, if it has one
4. multiply it times another 2 x 2 matrix.

Given a test covering the above material, you will be able to complete 70% of it correctly.

Pre-test:

$$A = \begin{vmatrix} 1 & 3 \\ 4 & 8 \end{vmatrix} \qquad \text{Find } A^T,\ A^{-1},\ A^2,\ \delta(A)$$

Information Sources:

Read Data Brief # 1	"Matrix Operations"
Read Data Brief # 2	"Multiplication of Matrices"
Read Data Brief # 3	"The Identity Matrix"
Read Data Brief # 4	"Inverse Matrices"
Read Data Brief # 5	"Determinants"
Read Data Brief # 6	"Sum Notation"

Data Brief # 1

As you have studied more and more sophisticated mathematics, you have had occasion to use more and more sophisticated kinds of numbers. You began with the set of counting numbers, 1, 2, 3, ... Then in oreder to make subtractions like 3 - 7 possible, the system was extended to the entire set of integers, 0, ±1, ±2, ±3, ... Next, in order to make it possible to divide any number by any nonzero number, rational numbers like 1/2, -2/3, -157/321, and 4/2 were invented. This did not bring you to the end of the story, for, in order that every positive number should have a square root, a cube root, a logarithm, etc., it was necessary to invent still more numbers: the infinite decimals or real numbers, such as 1.4142...., 3.1415928..., and 0.131313.... Finally, in order that negative numbers should also have square roots, and that such quadratic equations as

$$x^2 + x + 1 = 0$$

should have solutions, it was necassary to invent complex numbers like 3 + 2i, 1 + πi, etc.

Whenever there has seemed to be a good reason to do so, new sets of "numbers" have been invented. For instance, in inventing complex quantities, we began not with the quantities themselves but with a purpose: to find a system of numbers each of which has a square root. When one such invention has been made, it is not hard to realize that there is no reason to stop inventing.

It is easy to invent things that do not work, but hard to invent things that do work—easy to invent things that are useless, but hard to ;invent things that are useful. The same is true of the invention of new kinds of numbers. The hard thing is to invent useful kinds of numbers, and kinds of numbers "that work". Nevertheless, several more or less successful new kinds of numbers have been invented by mathematicians. At this time, you are going to study one of the most sucessful of these new kinds of numbers: the matrices.

Matrices are useful in almost every branch of science and engineering. A great number of the operations performed by the giant "electronic brains" are computed with matrices. Many problems in statistics are expressed in terms of matrices. Matrices come up in the mathematical problemsof economics. They are extremely important in the study of atomic physics; indeed, atomic physicists express almost all their problems in terms of matrices, and it would not be an exageration to say that the algebra of matrices is the language of atomic physics. Many other kinds of algebra, such as complex-number algebra and vector algebra, can be explained very easily in terms of matrices. So, in studying matrices, you will be studying one of the newest and most important, as well as one of the most interesting branches of mathematics.

Now let's take a look at a few simple examples.

Many a baseball fan, when he first opens the newspaper, refers to a tabulation similar to the following:

	G	AB	R	H
Aaron	68	280	52	109
Williams	52	194	29	60
Mantle	60	228	51	70
Lopez	63	241	38	72

If he is a Mantle fan, he looks at the entry in the third row and fourth column of numbers in order to learn hew many hits Mantle has thus far obtained during the season.

You will note that we have said "row" in speaking of a horizontal array, and "column" in speaking of a vertical array. Thus the third row is

60 228 51 70,

and the fourth column is

$$\begin{matrix} 109 \\ 60 \\ 70 \\ 72 \end{matrix}$$

An assembler of TV sets might have before him a table of the following sort:

	Model A	Model B	Model C
Number of tubes	13	18	20
No. of speakers	2	3	4

This table indicates the number of tubes and the number of speakers used in assembling a set of each model.

Omitting the row and column headings, let us focus our attention on the arrays of numbers in the last two examples:

$$\begin{matrix} 68 & 280 & 52 & 109 \\ 52 & 194 & 29 & 60 \\ 60 & 228 & 51 & 70 \\ 63 & 241 & 38 & 72 \end{matrix} \qquad \begin{matrix} 13 & 18 & 20 \\ 2 & 3 & 4 \end{matrix}$$

Such arrays of <u>entries</u> are called <u>mat[illegible]ces</u> (singular:

matrix). Thus a matrix is a rectangular array of entries appearing in rows and columns. Actually, the entries may be complex numbers, functions, and in appropriate circumstances even matrices themselves; however, with a few exceptions that will be clearly indicated, we shall confine our attention to the real numbers with which we are already familiar.

Some examples of matrices are the following:

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{2} \\ 3.1 & 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 1/2 & 1/4 & 1/8 \end{bmatrix} \qquad (1)$$

You will note here how square brackets $\left[\quad \right]$ are used in the mathematical designation of matrices.

.. great advantage of this notation is the fact that you can use it in handling large sets of numbers as single entities, thus simplifying the statement of complicated relationships.

The <u>order</u> of a matrix is g.ven by stating first the number of rows and then the number of columns in the matrix. Thus the oreders of the matrices in the foregoing examples (1) are respectively 2x3 (read "2 by 3"), 2x2, 4x1, 1x3. Generally a matrix that has m rows and n columns is called an m x n (read "m by n") matrix, or a matrix of order m x n.

If the number of rows is the same as the number of columns, then the matrix is <u>square</u>. Thus, given two linear equations in two unknowns.

$$2x + 3y = 7$$

$$x - 2y = 0$$

you observe that the coefficients of x and y constitute a square matrix:

$$\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix}.$$

When speaking of a square matrix (n x n), its order is often referred to as n rather than n x n. For example, the 2x2 matrix

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

is a square matrix of order 2, and the 3x3 matrix

Now let's take a look at a few simple examples.

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You will note here how square brackets $\begin{bmatrix} & \end{bmatrix}$ are used in the mathematical designation of matrices.

A great advantage of this notation is the fact that you can use it in handling large sets of numbers as single entities, thus simplifying the statement of complicated relationships.

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$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$$

is a square matrix of order 2, and the 3x3 matrix

$$\begin{bmatrix} -1 & 2 & 3 \\ 4 & -5 & 6 \\ 7 & 8 & -9 \end{bmatrix}$$

is a square matrix of order 3.

If the number of rows is 1, as in the fourth example in (1), above, the matrix is sometimes called a <u>row matrix</u> or a <u>row vector</u>. For example in terms of rectangular coordinates, a point in a plane might be designated by the row matrix $\begin{bmatrix} 2 & 3 \end{bmatrix}$, or a point in space by the row matrix $\begin{bmatrix} 2 & 3 & -1 \end{bmatrix}$.

Similarly, a <u>column matrix</u> or <u>column vector</u> is a matrix having just one column.

Read in your textbook at the top of page 544 about a "transpose."

Data Brief # 2

Thus far, we have defined and studied the addition and subtration of matrices and the multiplication of a matrix by a number. We still have not defined the product of two matrices. Since the formal definition is somewhat camplicated and may at first seem odd, let us look at a simple practical problem that will lead us to operate with two matrices in the way that we shall ultimately call multiplication.

In a previous section, the number of tubes and the number of speakers used in assembling TV sets of three different models were specified by a table:

	Model A	Model B	Model C
Number of tubes	13	18	20
Number of speakers	2	3	4

This array will be called the parts-per-set matrix.

Suppose orders were received in January for 12 sets of model A, 24 sets of model B, and 12 sets of model C; and in February for 6 sets of model A, 12 of model B, and 9 of model C. We can arrange the information in the form of a matrix:

	January	February
Model A	12	6
Model B	24	12
Model C	12	9

This will be called the sets-per-month matrix.

To determine the number of tubes and speakers required in each of the months for these orders, it is clear that we must use both sets of information. For instance, to compute the number of tubes needed in January, we multiply each entry in the 1st <u>row</u> of the parts-per-set matrix by the corresponding entry in the 1st <u>column</u> of the sets-per-month matrix , and then add the three products. Thus the number of tubes required in January is

$$13(12) + 18(24) + 20(12) = 828$$

To compute the number of speakers needed in January, we multiply each entry in the 2nd row of the parts-per-set matrix by the corresponding entry in the 1st column of the sets-per-month matrix and then add the products. Thus the number of speakers for January is

$$2(12) + 3(24) + 4(12) = 144$$

For February, first we multiply the entries from the 1st row of the parts-per-set matrix by the corresponding entries from the 2nd column of the sets-per-month matrix and add to determine the number of tubes; secondly, we multiply the entries from the 2nd row of the parts-per-set matrix by the corresponding entries from the 2nd column of the sets-per-month matrix and add to determine the number of speakers. Thus the number of tubes and speakers for February are, respectively,

$$13(6) + 18(12) + 20(9) = 474$$

and

$$2(6) + 3(12) + 4(9) = 84$$

We can arrange the four sums in an array, which we shall call the parts-per-month matrix:

	January	February
Number of tubes	828	474
Number of speakers	144	84

Can we now represent our "operation" in equation form? Yes.

$$\begin{bmatrix} 13 & 18 & 20 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 12 & 6 \\ 24 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 828 & 474 \\ 144 & 84 \end{bmatrix}$$

We have 'multiplied' the parts-per-set matrix by the set-per-month matrix to get what should be expected, the parts-per-month matrix!

Note that, in Equation (1), 828 equals the sum of the products of the entries in the 1st row of the left-hand factor by the corresponding entries in the first column of the right-hand factor. Likewise, 474 equals the sum of the products of the entries in the 1st row of the left-hand factor by the corresponding entries in the 2nd colimn of the right-hand factor, and so on. Consider the "product" matrix

$$\begin{bmatrix} 828 & 474 \\ 144 & 84 \end{bmatrix}$$

in the symbolic form,

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The subscripts indicate the row and column in which the entry appears; they also indicate ~~the row~~ and column of the two factor matrices that are combined to get that entry. Thus the entry a_{21} in the 2nd row and 1st column is found by adding the products formed when the entries in the 2nd row of the left-hand factor are multiplied by the corresponding entries in the 1st column of the right-hand factor. The most concise description of the process is: "Multiply row by column."

The description, "Multiply row by column," of the pattern in the foregoing simple practical problem serves as our guide in establishing the general rule for the multiplication of two matices. Very simply the rule is to multiply entries of a row by corresponding entries of a column and then add the products. Thus, given two matrices A and B, to find the entry in the i-th row and j-th column of the <u>product matrix</u> AB, multiply each entry in the i-th row of the left-hand factor A by the corresponding entry in the j-th column of the right-hand factor B, and then add all the resulting terms. Since there must be an entry in each row of the left-hand factor to match with each entry in a column of the right-hand factor, and conversely, it follows that the product is not defined unless the number of columns in the left-hand factor is equal to the number of rows in the right-hand factor. When the number of columns in the left-hand factor equals the number of rows in the right-hand factor, the matrices are <u>conformable for multiplication</u>.

A diagram can aid understanding; see Figure 1-1.

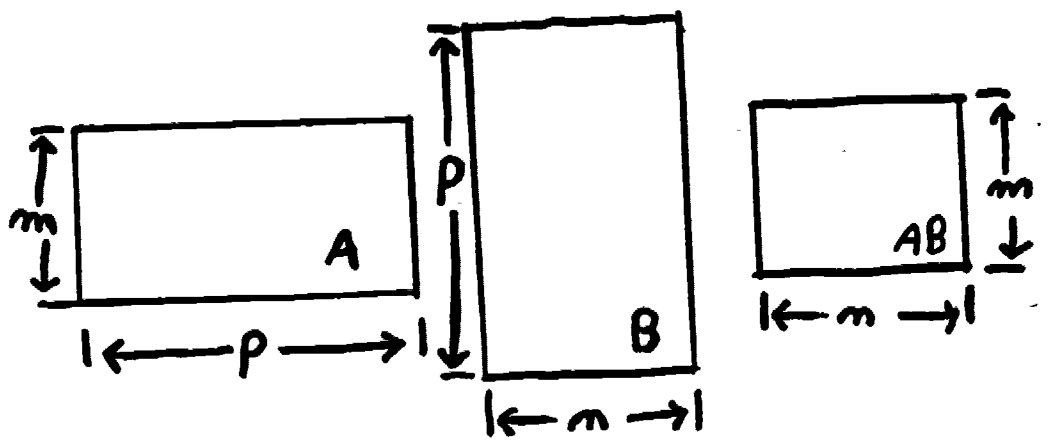


Figure 1-1. Matrices A and B that are conformable for multiplication. The number of columns of A must be equal to the number of rows of B. Then the product AB has the same number of rows as A and the same number of columns as B.

An entry in the product AB is found by multiplying each of the p entries in a row of A by the corresponding one of the p entries in the column of B and taking the sum; see Figure 1-2.

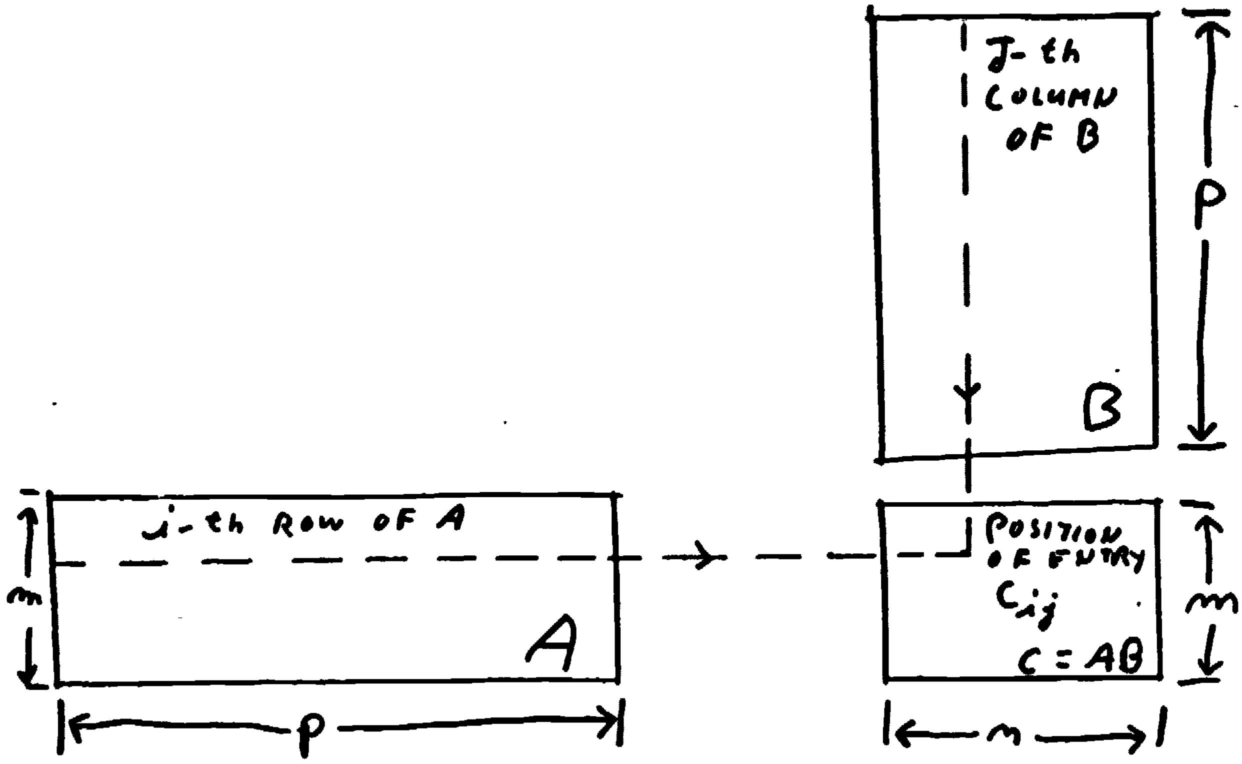


Figure 1-2. Determination of an entry in the product AB of matrices A and B that are conformable for multiplication.

Thus, for the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 4 & 1 \end{bmatrix}$$

to form the product AB, we compute as follows:

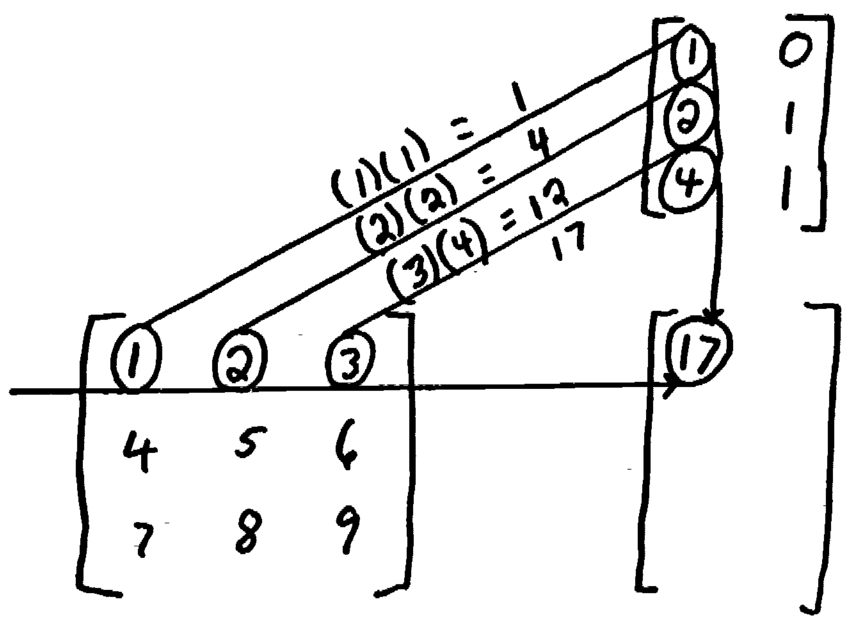


Diagram of matrix multiplication

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u & v \\ w & x \end{bmatrix} \qquad \begin{bmatrix} au + bw & \\ & \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u & v \\ w & x \end{bmatrix} \qquad \begin{bmatrix} au + bw & av + bx \\ & \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u & v \\ w & x \end{bmatrix} \qquad \begin{bmatrix} au + bw & av + bx \\ cu + dw & \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} u & v \\ w & x \end{bmatrix} \qquad \begin{bmatrix} au + bw & av + bx \\ cu + dw & cv + dx \end{bmatrix}$$

Data Brief # 3

You may be wondering if there is an identity element for the multiplication of matrices, namely a matrix that plays the same role as the number 1 does in the multiplication of real numbers. (For all real numbers a, $1 \times a = a = a \times 1$.) There is such a matrix, called a unit matrix, or the identity matrix for multiplication, and denoted by the symbol I. The matrix I_2, namely,

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

is called the unit matrix of order 2. The matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is called the unit matrix of order 3. In general, the unit matrix of order n is the square matrix

$$\begin{bmatrix} e_{ij} \end{bmatrix}_{n \times n}$$

such that $e_{ij} = 1$ for all $i=j$ and $e_{ij} = 0$ for all $i \neq j$ $(i = 1,2,\ldots,n;\ j = 1,2,\ldots,n)$. In general the theorem for the identity matrix states that

$$IA = A = AI.$$

For example,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

We now turn our attention to the problem of matrix division. (Instead of dividing by a number, Z, we will multiply by its reciprocal, 1/Z.) This problem arises when we seek to solve a matrix equation of the form

$$AX = C.$$

Let's look at a parallel equation concerning real numbers,

$$ax = c.$$

Each non-zero number a has a reciprocal $1/a$, which is often designated a^{-1}. Its defining property is $aa^{-1} = 1$. Since multiplication of real numbers is commutative, it follows that $a^{-1}a = 1$. Hence if a is a non-zero number, then there is a number b, called the multiplicative inverse of a, such that

$$ab = 1 = ba \qquad (b=a^{-1}).$$

Given an equation $ax = c$, where $a \neq 0$, the multiplicative inverse b enables us to find a solution of x; thus,

$$\begin{aligned} b(ax) &= bc \\ (ba)x &= bc \\ 1x &= bc \\ x &= bc. \end{aligned}$$

Now our question concerning division by matrices can be put in another way. If $A \in M$, is there a $B \in M$ for which the equation

$$AB = I = BA$$

is satisfied? We shall embloy the more suggestive notation A^{-1} for the inverse, so that our question can be restated: Is these an element A^{-1} M for which the equation

$$AA^{-1} = I = A^{-1}A$$

is satisfied?

From the fact that there is a multiplicative inverse for every real number except zero, you might wrongly infer a parallel conclusion for matrices.

Now let us try to find the inverse of the matrix designated A, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

and try to solve the equation $AX = I$.
If we let

$$X = \begin{bmatrix} p & q \\ r & s \end{bmatrix},$$

then we find that

$$AX = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} p & q \\ 0 & 0 \end{bmatrix}.$$

Hence, no matter what entries we take for X, we cannot have

$$AX = I$$

since the entry in the lower right-hand corner of AX is zero, and the entry in the lower right-hand corner of I is 1.

At this point you might be thinking that __no__ matrix has an inverse. But, note that $I \cdot I = I = I \cdot I$. This means that I is its own inverse, just as 1 is its own inverse among the real numbers.

Also, note that

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Thus the matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

has the inverse $A^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$.

Consequently, the equation $AX = B$ may be solved by the use of an inverse matrix in the case illustrated below.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Now multiply both sides of the equation by A^{-1}.

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} X = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X = \begin{bmatrix} 1/2 & 1 \\ 3/2 & 2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1/2 & 1 \\ 3/2 & 2 \end{bmatrix}$$

Data Brief # 5

It is the purpose now to develop a general method of determining the inverse of a 2×2 matrix. Instead of having specific numbers for entries, we let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The inverse will be represented by B, where

$$B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

If AB = I, then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This matrix equation may be written as four equations,

$$ap + br = 1 \quad (1) \qquad aq + bs = 0 \quad (3)$$

$$cp + dr = 0 \quad (2) \qquad cq + ds = 1 \quad (4)$$

Since we wish to find values for p, q, r, s, in terms of the real numbers a, b, c, and d, we multiply Equation (1) by d, Equation (2) by b, and then subtract. We obtain

$$adp - bcp = d$$

or

$$(ad - bc)p = d$$

Repeating this process, using appropriate pais of equations, we obtain

$$(ad - bc)q = -b, \quad (ad - bc)r = -c, \quad (ad - bc)s = a$$

Should it happen that $ad - bc = 0$, then it follows from the four equations, above, that $a = b = c = d = 0$, so that $A = 0$. But the zero matirx has no inverse, just as 0 has no inverse in regular algebra. In the beginning of this section we assumed that A did have an inverse, B; hence if $ad - bc = 0$ we have contradiction of this assumption. In other words, if A has an inverse, then $ad - bc \neq 0$.

Temporarily, let us denote the number $ad - bc$ by h. Now if $h \neq 0$ we may write

$$p = d/h, \quad q = -b/h, \quad r = -c/h, \quad s = a/h$$

Substituting these values appropriately in B, we obtain

$$B = \begin{bmatrix} d/h & -b/h \\ c/h & a/h \end{bmatrix} = 1/h \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In order to show that this matrix is the inverse of A, we check

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{d}{h} & \frac{-b}{h} \\ \frac{-c}{h} & \frac{a}{h} \end{bmatrix} = \begin{bmatrix} \frac{ad-bc}{h} & \frac{-ab+ab}{h} \\ \frac{cd-cd}{h} & \frac{-bc+ad}{h} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

By the same procedure it can be shown that BA = I. The fact that this relationship follows from the relationship AB = I is quite significant. While the definition of the inverse demands the existance and equality of what are called left and right inverses, we have showb that for 2 x 2 matrices the existance of one implies the existance of one implies the existance of the other and that if they exist, then they are the same. Since the multiplication of matrices is not generally commutative, you might have expected otherwise.

Formally the theorm for matrix inverses states that,

If and only if the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse, then $ad - bc \neq 0$.

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = X$ then $\delta(X) = ad - bc$ and is called the <u>determinant</u> of X.

Data Brief # 6

If $S = x_1 + x_2 + \dots x_p$, It may also be

written as $S = \sum_{j=1}^{p} x_j$

For example, $\sum_{j=1}^{p} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 55$

The formula for the sum of the firs' ~ positive integers is,

$1 + 2 + 3 + \dots + p = \frac{p(p+1)}{2}$ It can also be

expressed as $\sum_{j=1}^{p} j = \frac{p(p+1)}{2}$

In this notation, the sum $a_{31}b_{14} + a_{32}b_{24} + \dots + a_{3p}b_{p4}$

is expressed as $\sum_{j=1}^{p} a_{3j}b_{j4}$

You might recognize this as the element in the third row and fourth column of the matrix AB.

(Activity # 6 is a lecture)

Activity # 1

P. 546
In your text book, do all odd problems 1-21, and 24, 26.

Activity # 2

P. 551
In your text book, written exercises, # 1,3,5,7,9

Multiply the two matrices below.

$$\begin{bmatrix} 1 & 3 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}$$

Prove that in matrix algebra, multiplication is not commutative.

Set the

Set up the proof that multiplication is assodiative

Activity # 3

Find x such that when A is multiplied times B the product is the identity matrix I.

That is $AB = I$

1. $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} x & -5 \\ -1 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 3 & x \\ -2 & 3 \end{bmatrix}$

3. $A = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} x & -1 \\ -1 & 1 \end{bmatrix}$

4. $A = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 5 & -4 \\ -4 & x \end{bmatrix}$

5. $\begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -x & -14x & 7x \\ 0 & 1 & 0 \\ x & 4x & -2x \end{bmatrix} = I$

1. Which of the following pairs of elements are inverses of one another?

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -1 \\ 2 & -3 \end{bmatrix}$ and $\begin{bmatrix} 3 & -1 \\ 2 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 4 \\ 6 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -4 \\ -6 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} -5 & 7 \\ 0 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

2. The matrices $\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix}$ and $\begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}$ are inverses of one another. Are their squares also inverses? Their transposes?

1. For each of the following matrices, determine whether the inverse exists; if it does exist, find it.

a) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ c) $\begin{bmatrix} -3 & 7 \\ 9 & 21 \end{bmatrix}$ d) $\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}$

e) $\begin{bmatrix} -2 & 0 \\ 3 & 4 \end{bmatrix}$ f) $\begin{bmatrix} 2 & a \\ 0 & -7 \end{bmatrix}$ g) $\begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$

2. Determine those values of x for which the matrix has no inverse.

a) $\begin{bmatrix} x^2 & 1 \\ 1 & x \end{bmatrix}$ b) $\begin{bmatrix} x^3 & x \\ 0 & 1 \end{bmatrix}$ c) $\begin{bmatrix} x+2 & 0 \\ x^4 & x-1 \end{bmatrix}$ d) $\begin{bmatrix} x^2 & x-1 \\ 2 & 3 \end{bmatrix}$

In each exercise using the matrices below, show that; $\delta(AB) = \delta(A)\,\delta(B)$ and also that $\delta(B^{-1}) = \dfrac{1}{\delta(B)}$

3. $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

4. $A = \begin{bmatrix} t^2 & 1 \\ -1 & t \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

5. $A = \begin{bmatrix} x & x^2 \\ x^3 & x^4 \end{bmatrix}$ $B = \begin{bmatrix} x & -x \\ 3 & 4 \end{bmatrix}$

---

6. Using matrix A in exercise 3 above show that $\delta(tA) = t^2\delta(A)$

7. If $A = \begin{bmatrix} x & 1 \\ x^2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}$ SHOW THAT $\delta(B^{-1}AB) = \delta(A)$.

8. If X^t is the transpose of X, and $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ show that $\delta(X) = \delta(X^t)$ and conclude that $\delta(AA^t) \geqslant 0$.

SAHUARITA HIGH SCHOOL

CAREER

CURRICULUM

PROJECT

I Don't Function Right!

Information Sources:

Read Data Brief # 1	"Set Notation"
Read Data Brief # 2	"Functions"
Read Data Brief # 3	"Graphing a function"
Read Data Brief # 4	"Constant and Linear Functions"
Read Data Brief # 5	"The Absolute-value Function"
Read Data Brief # 6	"Inversion"

Data Brief # 1

A set is a collection of objects - not necessarily material objects - described in such a way that there is no doubt as to whether a particular object does or does not belong to the set.

Usually capital letters are used to designate sets.

A set may be described by listing its elements within braces, as

$$A = \{1, 2, 3, 4\}$$

or by using the set-builder notation, as

$$A = \{x\colon x \text{ is a positive integer and } x < 5\}.$$

(this should be read "A is the set of all x such that x is a positive integer and x is less than 5.")

The Greek letter $\in$ (epsilon) is used to indicate that an element belongs to a given set, as $2 \in A$. (Read this, "2 is an element of the set A" or "2 belongs to the set A.")

The intersection of two sets A and B, written $A \cap B$, is the set of all elements that belong to A and also belong to B: $A \cap B = \{x\colon\ x \in A \text{ and } x \in B\}$.

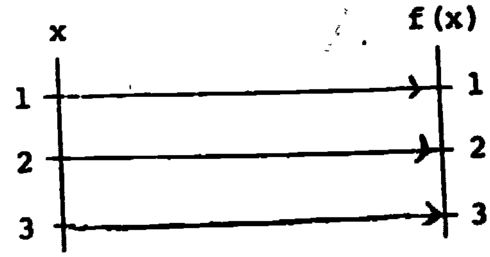
The union of two sets A and B, written $A \cup B$, is the set of all elements that belong to A or to B or to both: $A \cup B = \{x\colon\ x \in A \text{ or } x \in B\}$.

Domain: The set whose elements may serve as replacements for a variable, x.

Range: The values of $f(x)$ when you pick a domain for x.

For a set of ordered pairs, {(a,b), (3,6), (#,@), (x,y)}, the set of all the first elements is the domain; a, 3, #, x. The set of all the second elements is the range; b, 6, @, y.

In the open sentence, $y = f(x) = x$, does each element of the domain take you into one and only one element in the range?



If the domain of this function is all the real numbers, then the range is also all the real numbers.

Data Brief # 2

We frequently hear people say, "One function of the Police Department is to prevent crime," or "Several of my friends attended a social function last night," or "My car failed to function when I tried to use it." In mathematics we use the word "function" somewhat differently than we do in ordinary canversation; as you have probably learned in your previous study, we use it to denote a certain kind of association or dorrespondence between the members of two sets.

For instance, we note such an association between the number of feet a moving object travels and the difference in clock readings at two separate points in its journey; between the price of eggs and the cost of making a cake; between the length of a steel beam and its temperature. Additional examples of such associations accour in geometry, where, for example, we have the area of the circumference of a circle associated with the length of its radius.

In all of these examples, regardless of their nature, there seems to be the natural idea of a direct connection of the elements of one set to those of another; the set of distances to the set of times, the set of lengths to the set of thermometer readings, etc. It seems natural, therefore, to abstract from these various cases this idea of association or correspondence and examine it more closely.

Let us start with some very simple examples. Suppose we take the numbers 1, 2, 3, and 4, and with each of them associate the number twice as large: with 1 we associate 2, with 2 we associate 4, with 3 we associate 6, and with 4 we associate 8. An association such as this is called a function, and the set $\{1, 2, 3, 4\}$ with which we started is called the domain of the function. We can represent this association more briefly if we use arrows instead of words: $1 \longrightarrow 2$, $2 \longrightarrow 4$, $3 \longrightarrow 6$, $4 \longrightarrow 8$.T There are, of course, many other functions with the same domain; for example, $1 \longrightarrow 2$, $2 \longrightarrow 1$, $3 \longrightarrow 2$, $4 \longrightarrow 5$.

It happens that these two examples deal with numbers, but there are many functions which do not. A map, for instance, associates each point on some bit of terrain with a point on a piece of paper; in this case, the domain of the function is a geographical region. We can, indeed, generalize this last example, and think of any function as a mapping; thus, our first two examples map numbers into numbers, and our third maps points into points.

What are the essential features of each of these examples? First, we are given a set, the domain. Second, we are given a rule of some kind which associates an object of some sort with each element of the domain, and, third, we are given some idea of where to find this associated object. Thus, in the first example above, we know that if we start with a set of real numbers, and double each, the place to look for the result is in the set of all real numbers. To take still another example, if the domain of a function is the set of all real numbers, and the rule is "take the square root," then the set in which we must look for the result is the set of complex numbers. We summarize this discussion in the following definition:

Definition 1-1. If with each element of a set A there is associated in some way exactly one element of a set B, then this association is called a function from A to B.

It is common practice to represent a function by the letter "f" (other letters such as "g" and "h" will also be used.) If x is an element of the domain of a function f, then the object which f associates with x is denoted $f(x)$ (read "the value of f at x" or simply "f at x" or "f of x"); $f(x)$ is called the image of x. Using the arrow notation of our examples, we can represent this symbolically by

$$f: x \longrightarrow f(x)$$

(read "f takes x into $f(x)$"). This notation tells us nothing about the function f or the element x; it is merely a restatement of what "f(x)" means.

The set A mentioned in Definition 1-1 is, as has been stated, the domain of the function. The set of a'l objects onto which the function maps the element of A is called the range of the function; in set notation,

$$\text{Range of } f = \{f(x) : x \in A\}.$$

The range may be the entire set B mentioned in the definition, or may be only a part thereof, but in either case it is included in B.

It is often helpful to illustrate a function as a mapping, showing the elements of the domain and the range as points and the function as a set of arrows from the points that represent elements of the domain to the points that represent elements of the range, as in figure 1-1a. Note that, as a consequence of Definition 1-1,

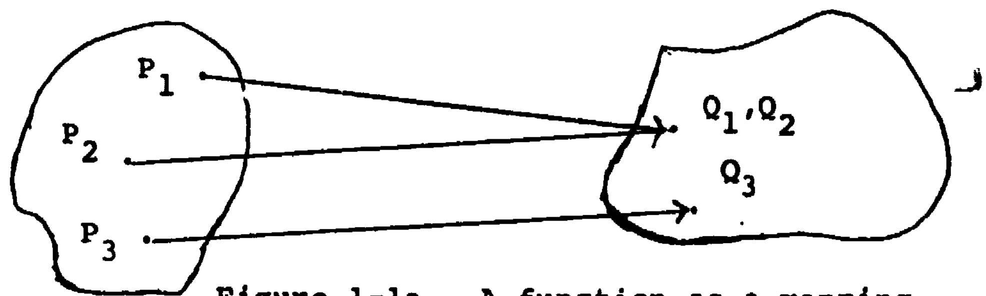


Figure 1-1a. A function as a mapping.

to each element of the domain there corresponds one and only one element of the range, If this condition is not met, as in figure 1-1b, then the mapping pictured is not a function. In terms of the pictures, a mapping is not a function if two arrows start from one point; whether two arrows go to the same point, as in Figure 1-1a, is immaterial in the definition. This requirement, that each element of the domain be mapped onto one and only one element of the range, ~~may~~ ~~be arbitrary~~ seem arbitrary, but it turns out, in practice to be extremely convenient.

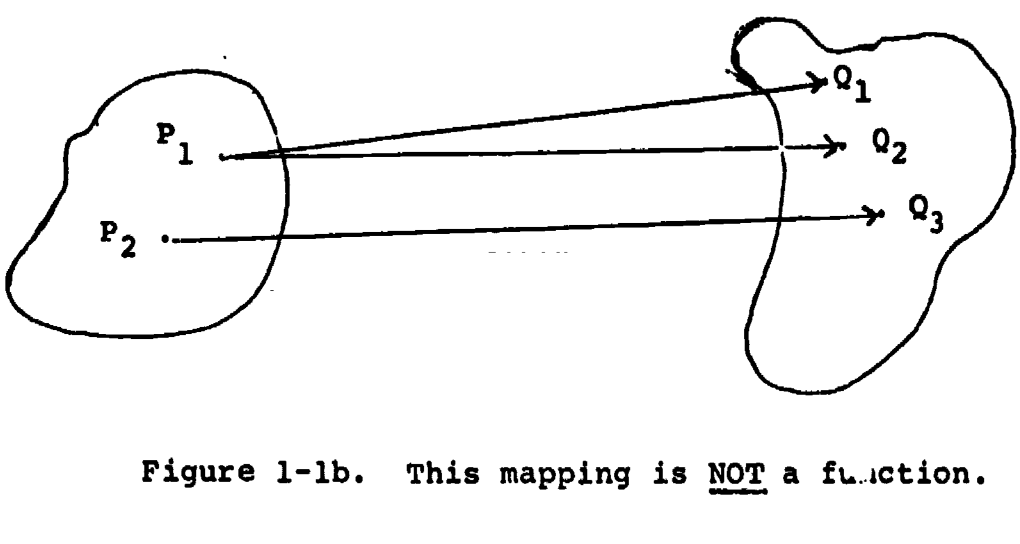


Figure 1-1b. This mapping is NOT a function.

Since most of the functions we will be dealing with will have domains and ranges in the set of real numbers, it is convenient to represent the domain by a set of points on a number line and the range as a set of points on another number line, as in figure 1-1d.

Specifically, consider the function f, discussed earlier, which takes each element of the set 1, 2, 3, 4 into the number twice as great. The range of this function is 2, 4, 6, 8 and f maps its domain onto its range as shown in Figure 1-1d. We note that, in this case, the image of the element x of the domain of f is the element 2x; hence we may write, in this instance, $f(x) = 2x$, amd f is completely specified by the notation

$$f: x \longrightarrow 2x, \; x = 1, 2, 3, 4.$$

In this case, the way in which f maps its domain onto its range is completely specified by the formula $f(x) = 2x$. Most of the functions which we shall consider can similarly be described by appropriate formulas. If, for example, f is the function that takes each number into its square, then it takes 2 into 4 (that is, $f(2) = 4$), it takes -3 into 9 (that is $f(-3) = 9$), and in general, it takes any real number x into x^2. Hence, for this function, $f(x) = x^2$, we may write $f: x \longrightarrow x^2$. The formula $f(x) = x^2$ defines this function f, and to find the image of any element of the domain, we can merely substitute in this formula; thus, if $a - 3$ is a real number, then $f(a-3) = (a-3)^2 = a^2 - 6a + 9$. Similarly, if we know that a function f has $f(x) = 2x - 3$ for all $x \in R$ (we use R to represent the set of real numbers) then we can represent f in our mapping notation as f: x 2x - 3, and to x in the expression 2x - 3

notation as $f: x \longrightarrow 2x - 3$, and to find the image of any real number we need only substitute it for x in the expression $2x - 3$; thus $f(5) = 2(5) - 3 = 7$, $f(\sqrt{2}) = 2\sqrt{2} - 3$, and if $k + 2$ is a real number, then

$$f(k + 2) = 2(k + 2) - 3 = 2k + 1.$$

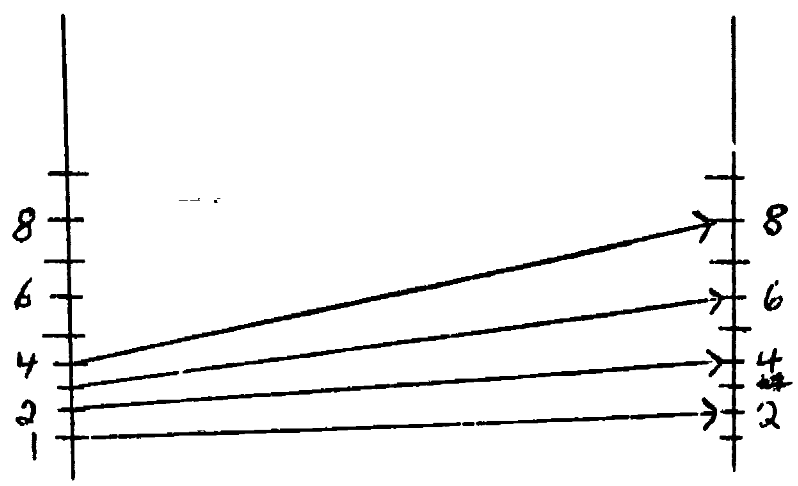


Figure 1-1d. $f: x \longrightarrow 2x, \; x = 1, 2, 3, 4$

Strictly speaking, a function is not completely described unless its domain is specified. In dealing with a formula, however, it is a common and convenient practice to assume, if no other information is given, that the domain includes all real numbers that yield real numbers when substituted in the formula. For example, if nothing further is said, in the function $f: x \dashrightarrow 1/x$, the domain is assumed to be the set of all real numbers ~~that~~ except 0; this exception is made because 1/0 is not a real number. Similarly, if f is a function such that $f(x) = \sqrt{1 - x^2}$, we assume, in the absence of other information, that the domain is $\{x: -1 < x \le 1\}$, that is, the set of all real numbers from -1 to +1 inclusive, since only these real numbers will give us real square roots in the expression for $f(x)$. When a function is used to describe a physical situation, the domain is understood to include only those numbers that are physically realistic. Thus if we are describing the volume of a balloon in terms of the length of its radius, $f: r \dashrightarrow V$, the domain would include only positive numbers.

Another way of looking at a function, which may help you to understand this section, is to think of it as a machine that processes elements of its domain to produce elements of its range. The machine has an input and an output; if an element x of its domain is fed on a tape into the machine, the element $f(x)$ of the range will appear as the output, as indicated in Figure 1-1e.

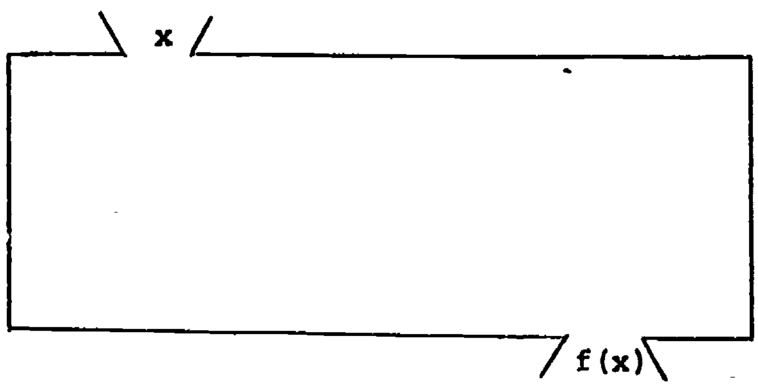


Figure 1-1e
A representation of a function as a machine.

A machine can only be set to perform a predetermined task. It cannot exercise judgement, make decisions, or modify its instructions. A function machine f must be set so that any particular input x always results in the same output $f(x)$; if the element x is not in the domain of f, the machine will jam or refuse to perform. Some machines - - notably computing machines - - actually do work in almost exactly this way.

Data Brief # 3

A graph is a set of points. If the set consists of all points whose coordinates (x,y) satisfy an equation in x and y, then the set is said to be the graph of that equation. If there is a function f such that, for each point (x,y) of the graph, and for no other points, we have $y = f(x)$, then we say that the graph is the graph of the function f. The graph is perhaps the most intuitively illuminating representation of a function; it conveys at a glance much important information about the function. The function $x \longrightarrow x^2$, (when there is no danger of confusion, we sometimes omit the name of a function, as "f" in $f:x \longrightarrow x^2$) has the parabolic graph shown in Figure 1-2a. We can look at the parabola and get a clear intuitive idea of what the function is doing to the elements of its domain. We can moreover, usually infer from the graph any limitations on the domain and the range. Thus,

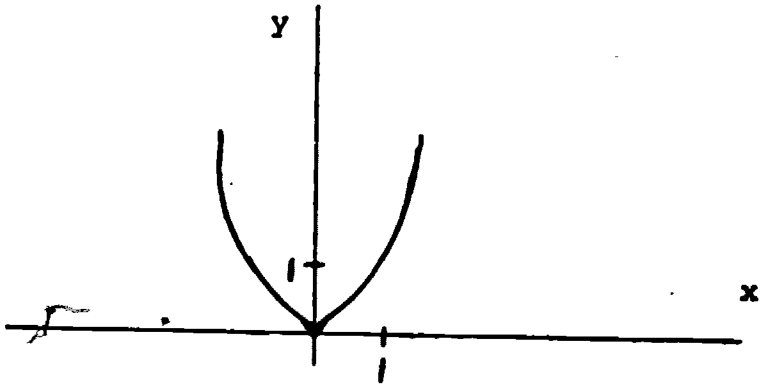


Figure 1-2a.
Graph of the function $f:x \longrightarrow x^2$

it is clear from Figure 1-2a, that the range of the function there graphed includes only non-negative numbers, and in the function $f:x \longrightarrow \sqrt{25 - x^2}$ graphed in Figure 1-2b, the domain $\{x: -5 \le x \le 5\}$ and range $\{y: 0 \le y \le 5\}$ are easily determined, as shown by the heavy segments on the x-axis and y-axis respectively.

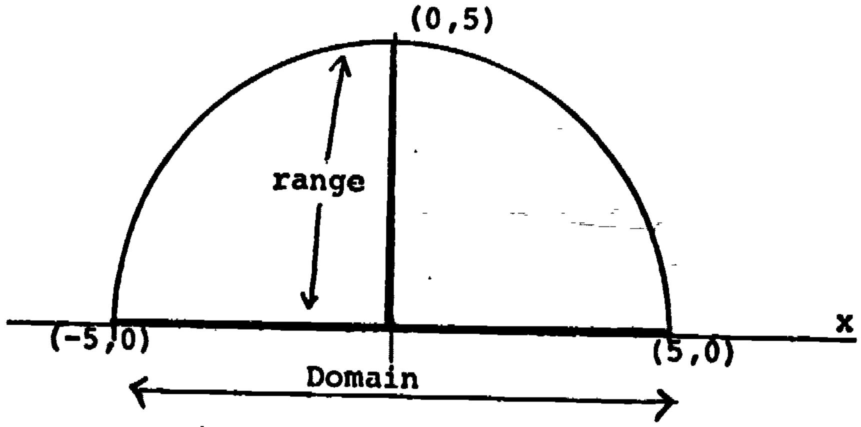


Figure 1-2b.
Graph of the function $f:x \longrightarrow \sqrt{25-x^2}$

Another illustration: the function

$$f: x \longrightarrow x/2, \quad 2 < x \le 6$$

has domain $A = \{x: 2 < x \le 6\}$ and range $B = \{f(x) : 1 < f(x) \le 3\}$. In this case we have used open dots at 2 on the x-axis and at 1 on the y-axis to indicate that these numbers are not elements of the domain and range respectively. See Figure 1-2c.

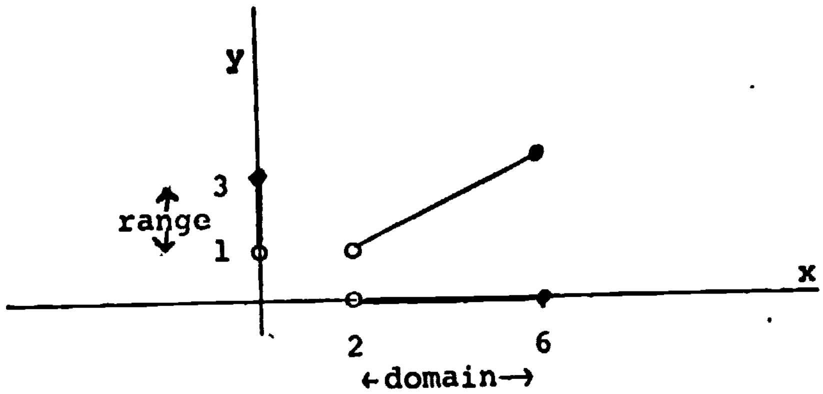


Figure 1-2c.
Graph of the function $f: x \longrightarrow x/2, \quad 2 < x \le 6$.

As might be expected, not every possible graph is the graph of a function. In particular, Definition 1-1 requires that a function map each element of its domain onto <u>only one</u> element of its range. In the language of the graphs, this says that only one value of y can correspond to any value of x. if, for example, we look at the graph of the equation $x^2 + y^2 = 25$, shown in Figure 1-2d, we can see that there are

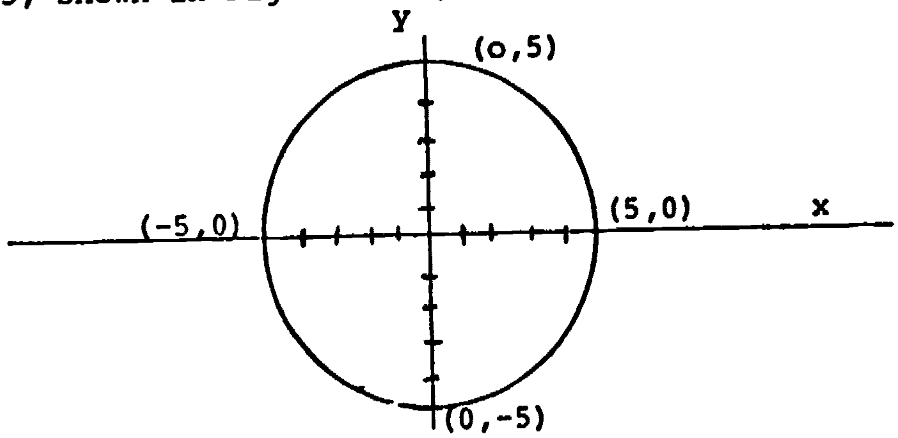


Figure 1-2d.
Graph of the set $S = \{(x,y) : x^2 + y^2 = 25\}$.

many instances in which one value of x is associated with two values of y, contrary to the definition of <u>function</u>. To give a specific example, if $x = 3$, we have both $y = 4$ and $y = -4$; each of the points $(3,4)$ and $(3,-4)$ is on the graph. Hence this is not the graph of a function. We can, however, break it into two pieces, the graph of $y = \sqrt{25 - x^2}$ and the graph of $y = -\sqrt{25 - x^2}$ (This makes the points $(-5,0)$ and $(5,0)$ do double duty), each of which is the graph of a function.

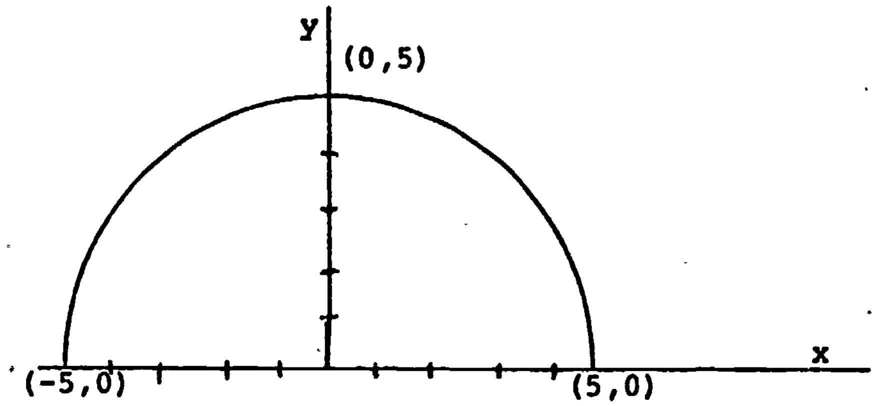


Figure 1-2e
Graph of $y = \sqrt{25 - x^2}$

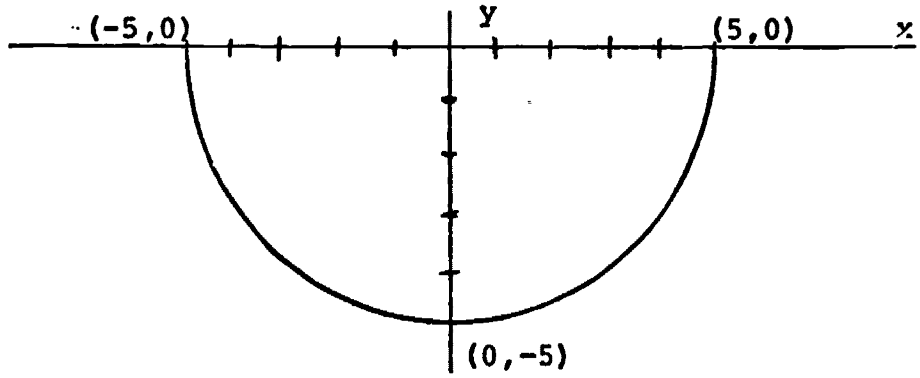
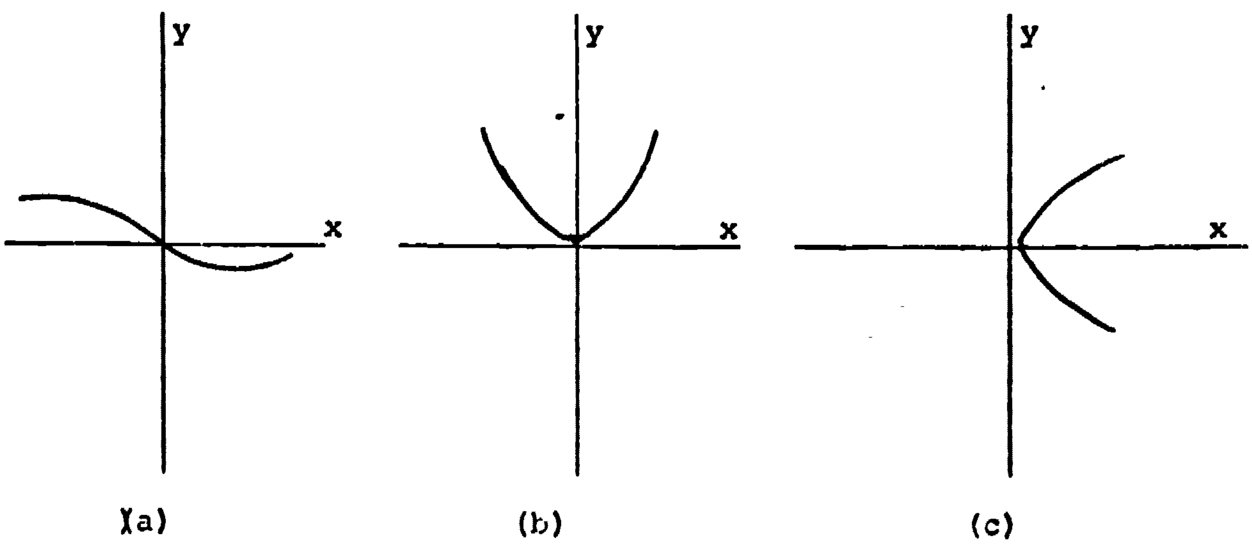


Figure 1-2f.
Graph of $y = -\sqrt{25 - x^2}$

If, in the xy-plane, we imagine all possible lines which are parallel to the y-axis, and if any of these lines cuts the graph in more than one point, then the graph defines a relation that is not a function. Thus, in figure 1-2g, (a) depicts a function, (b) depicts a function, but (c) does not depict a function.



Data Brief # 4

We have intoduced the general idea of function, which is a particular kind of an association of elements of one set with elements of another. We have also interpreted this idea graphically for functions which map real numbers into real numbers. In the previous sections general ideas were presented and specific ideas were only used for examples. In this section some specific functions will be studied.

Let us think of a man walking north along a long straight road at the uniform rate of 2 miles per hour. At some particular time, say time $t = 0$, this man passed the milepost located one mile north of baseline road. An hour befor this which we shall call time $t = -1$, he passed the milepost located one mile south of Baseline Road. An hour after time $t = 0$, at time $t = 1$, he passed the milepost located three miles north of Baseline Road. In order to form a convenient mathematical picture of man's progress, let us consider miles north of Baseline Road as positive and miles south of as negative. Thus the man passed milepost -1 at time $t = -1$, milepost 1 at time $t = 0$, and milepost 3 at time $t = 1$. Using an ordinary set of coordinate axes let us plot his position, as indicated by the mileposts, versus time in hours. This gives us the graph shown in Figure 1-3a.

In t hours the man travels $2t$ miles. Since he is already at milepost 1 at time $t = 0$, he must be at milepost $2t + 1$ at time t. This pairing of numbers is an example of a <u>linear</u> function.

Now let us plot the man's <u>speed</u> versus time. For all values of t during the time he is walking his speed is 2 miles per hour. This is graphed in Figure 1-3b.

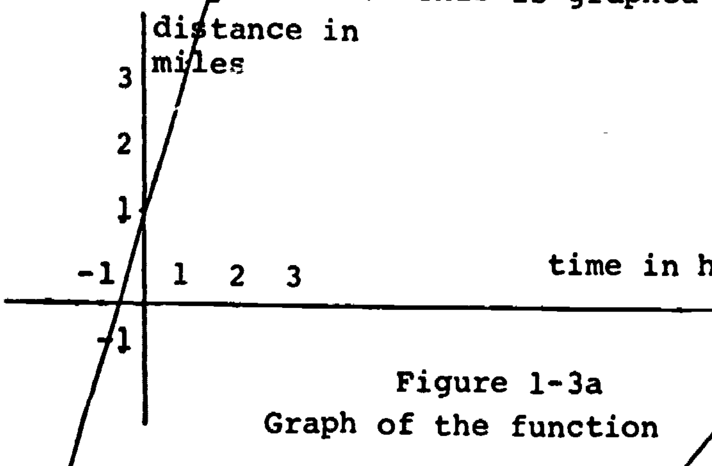
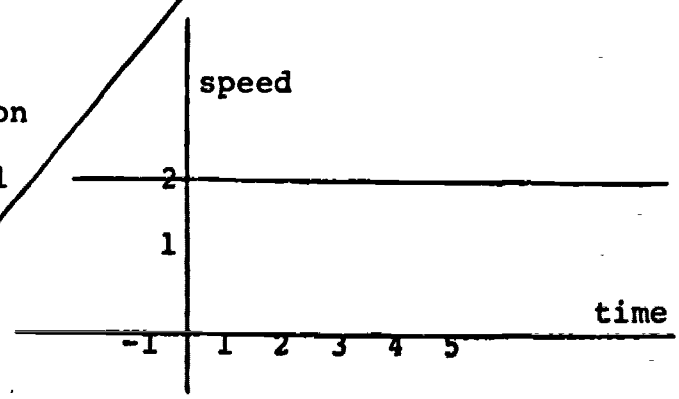


Figure 1-3a
Graph of the function

$f: t \longrightarrow d = 2t + 1$



graph of the function
$g: t \longrightarrow s = 2$

Figure 1-3b

When $t = -1$ his speed is 2, when $t = 0$ his speed is 2, etc.; with each number t we associate the number 2. This mapping, in which the range contains only the one number 2, is an example of a constant function.

DEFINITION. If with each real number x we associate one fixed number c, then the resulting mapping,

$$f: x \longrightarrow c,$$

is called a constant function.

The discussion of constant function can be disposed of in a few lines. The function we just mentioned, for example, is the constant function $g: t \longrightarrow 2$. The graph of any constant function is a line parallel to the horizontal x-axis. Constant functions are very simple, but they occur over and over again in mathematics and science and are really quite important. A well lnown example from physics is the magnitude of the attraction of gravity, which is usually taken to be constant over the surface of the earth-- though, in this age, we must recognize the fact that the attraction of gravity varies greatly throughout space.

The functions we examine next also occur over and over again in mathematics and science and are considerably more interesting than the constant functions. These are the linear Functions. Since you have worked with these functions before, we can begin at once with a formal definition.

DEFINITION. A function f defined on the set of all real numbers is called a linear function if there exist real numbers m and b, with $m \neq 0$, such that

$$f(x) = mx + b.$$

Example 1. The function $f: x \longrightarrow 2x + 1$ is a linear function. Here $f(0) = 1$, $f(1) = 3$, $f(-1) = -1$. This function was described earlier in this section in terms of t, with $f(t) = 2t + 1$. Its graph can be found in Figure 1-3a.

We note that the graph in Figure 1-3a appears to be a straight line. The graphs of all linear functions are straight lines.

An important property of any straight line segment is its slope, defined as follows.

DEFINITION. The slope of the line segment from the point $P(x_1, y_1)$ to the point $Q(x_2, y_2)$ is the number

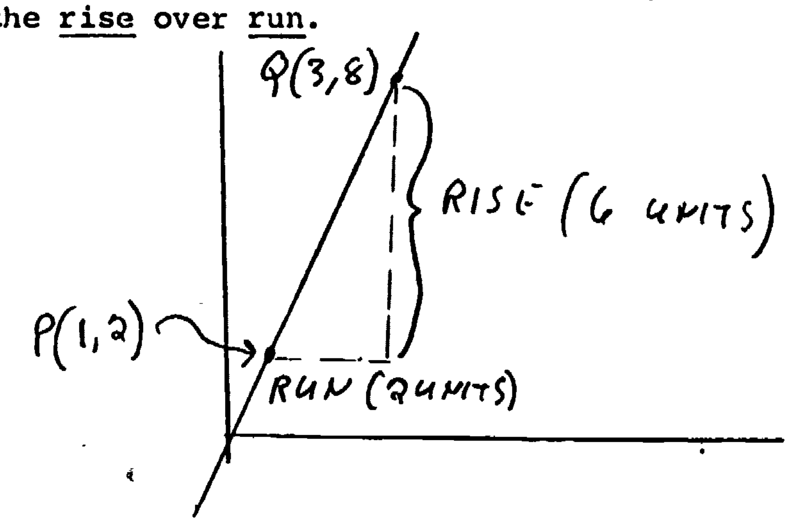
$$\frac{y_2 - y_1}{x_2 - x_1}$$

provided $x_1 \neq x_2$. If $x_1 = x_2$, the slope is not defined.

Note that $\dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{y_2 - y_1}{x_2 - x_1}$

So that it is immaterial which of the two points P or Q we take first.

What about the geometric meaning of the slope of a segment? Suppose, for the sake of definiteness, we consider the segment joining P(1,2) and Q(3,8). By our definition, the slope of this segment is 3, since $(8 - 2)/(3 - 1) = 3$. Note that this is the vertical distance from P to Q divided by the horizontal distance from P to Q, or, in more vivid language, the rise over run.



Let us think of the segment PQ as running from left to right, wo that the run is positive. If the segment rises, then the "rise" is positive and the slppe is positive. If the segment falls than the "rise" is negative and the slope is therefore negative. The steeper the segment, the larger the absolute value of the slope,

It has been stated that slope is not defined if $x_1 = x_2$. In this case the segment lies on a line parallel to the y-axis. It is important to distinguish this situation from the case $y_1 = y_2$ in which the line has a slope and it is 0.

Note that lines having zero slope, that is lines parallel to the x-axis, are graphs of constant functions. On the other hand, lines for which no slope is defined, that is lines parallel to the y-axis, cannot be graphs of any functions because, with one value of x, the graph associates more than one value--in fact, all real values.

If a line is the graph of a linear function $f: x \quad mx + b$, then for any x_1 and x_2, $x_1 \neq x_2$, the slope of the segment joining
joining $(x_1, F(x_1))$ AND $(x_2, F(x_2))$ is by definition

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} =$$

$$\frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = m$$

Data Brief # 5

A function of importance in many branches of mathematics is the absolute-value function, $f: x \longrightarrow |x|$ for all $x \in R$. The absolute value of a number describes the size, or magnitude, of the number, without regard to its sign; thus, for example $|2| = |-2| = 2$ (read "$|2|$" as "the absolute value of 2"). A common definition of $|x|$ is

<u>Definition</u> <u>1-5</u>.

$$|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0. \end{cases}$$

If a line is the graph of a linear function $f: x \quad mx + b$, then for any x_1 and x_2, $x_1 \neq x_2$, the slope of the segment joining
joining $(x_1, F(x_1))$ AND $(x_2, F(x_2))$ is by definition

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} =$$

$$\frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1} = m$$

Data Brief # 5

A function of importance in many branches of mathematics is the absolute-value function, $f: x \longrightarrow |x|$ for all $x \in R$. The absolute value of a number describes the size, or magnitude, of the number, without regard to its sign; thus, for example $|2| = |-2| = 2$ (read "$|2|$" as "the absolute value of 2"). A common definition of $|x|$ is

Definition 1-5.

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0. \end{cases}$$

You should be able to see, from the first definition of this function given above, that this graph consists of the origin, the part of the line $y = x$ that lies in Quadrant I, and the part of the line $y = -x$ that lies in Quadrant II.

There are two important theorems about absolute values.

Theorem 1-1. For any two real numbers a and b, $|ab| = |a| \cdot |b|$.

Proof: $|a| \cdot |b| = \sqrt{a^2} \cdot \sqrt{b^2} = \sqrt{a^2b^2} = \sqrt{(ab)^2} = |ab|$.

Theorem 1-2. For any two real numbers a and b, $|a + b| \leq |a| + |b|$.

Proof: By Definition 1-6, Theorem 1-2 is equivalent to

$$\sqrt{(a + b)^2} \leq \sqrt{a^2} + \sqrt{b^2}, \qquad (1)$$

which is equivalent to

$$a^2 + 2ab + b^2 \leq a^2 + 2\sqrt{a^2}\sqrt{b^2} + b^2,$$

and hence to $\quad 2ab \leq 2\sqrt{a^2}\sqrt{b^2}$

or $\quad ab \leq \sqrt{a^2}\sqrt{b^2}. \qquad (2)$

Now equation (2) is easy to prove. If a and b have opposite signs, then $ab < 0$ and (2) holds with the $<$ sign. Otherwise,

we have $\quad ab = \sqrt{a^2}\,\sqrt{b^2}$.

Hence in any case $\quad ab \leq \sqrt{a^2}\,\sqrt{b^2}$,

and therefore (1) holds. q.e.d.

Thus, for example, $|(-2)(3)| = |-6| = 6 = 2 \cdot 3 = |-2| \cdot |3|$, $|(-2) + (3)| = 1 < 5 = 2 + 3 = |-2| + |3|$, and $|(-2) + (-3)| = 5 = 2 + 3 = |-2| + |-3|$.

Data Brief # 6

Quite frequently in science and in everyday life we encounter quantities that bear a kind of reciprocal relationship to each other. With each value of the temperature of the air in an automobile tire, for example, there is associated one and only one value of the pressure of the air against the walls of the tire. Conversely, with each value of the pressure there is associated one and only one value of the temperature. Two more examples, numerical ones, will be found below.

Suppose that f is the function $x \longrightarrow x + 3$ and g is the function $x \longrightarrow x - 3$. Then the effect of f is to increase each number by 3, and the effect of g is to decrease each number by 3. Hence f and g are reciprocally related in the sense that each undoes the effect of the other. If we add 3 to a number and then subtract 3 from the result we get back to the original number. In symbols

$$(gf)(x) = g(f(x)) = g(x + 3) = (x + 3) - 3 = x.$$

Similarly,

$$(fg)(x) = f(g(x)) = f(x - 3) = (x - 3) + 3 = x.$$

As a slightly more complicated example we may take

$$f: x \longrightarrow 2x - 3 \text{ and } g: x \longrightarrow \frac{x + 3}{2}.$$

Here f says "Take a number, double it, and then subtract 3." To reverse this, we must add three and then divide by 2. This is the effect of the function g. In symbols,

$$(gf)(x) = g(f(x)) = g(2x - 3) = \frac{(2x - 3) + 3}{2} = x.$$

Similarly,

$$(fg)(x) = f(g(x)) = f\left(\frac{x + 3}{2}\right) = 2\frac{x + 3}{2} - 3 = x.$$

In terms of our representation of a function as a machine, the g machine in each of these examples is equivalent to the f machine running backwards; each machine then undoes what the other does, and if we hook up the two machines in tandem, every element that gets through both will come out just the same as it originally went in.

We now generalize these two examples in the following definition of inverse functions.

Definition 1-8. If f and g are functions so related that $(fg)(x) = x$ for every element x in the domain of g and $(gf)(y) = y$ for every element y in the domain of f, then f and g are said to be inverses of each other. In this case both f and g are said to have an inverse, and each is said to be an inverse of the other.

As a further example of the concept of inverse functions let us examine the functions $f: x \longrightarrow x^3$ and $g: x \longrightarrow \sqrt[3]{x}$. In this case

$$(fg)(x) = f(g(x)) = f(\sqrt[3]{x}) = (\sqrt[3]{x})^3 = x$$

and

$$(gf)(x) = g(f(x)) = g(x^3) = \sqrt[3]{x^3} = x$$

for all $x \in R$.

If a function f takes x into y, that is, if $y = f(x)$, then an inverse g of f must take y right back into x, that is, $x = g(y)$. If we make a picture of a function as a mapping, with an arrow extending from each element of the domain to its image, as in Figure 1-6a, then to draw a picture of the inverse function we need merely reverse the arrows, as in Figure 1-6b.

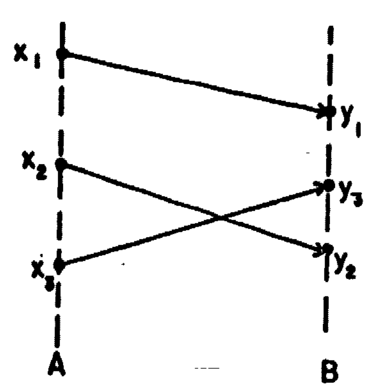


Figure 1-6a. A function.

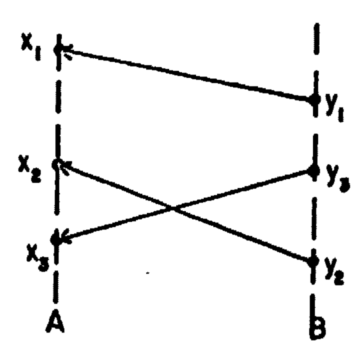


Figure 1-6b. Its inverse.

We can take any mapping, reverse the arrows in this way, and obtain another mapping. The important question for us, at this point, is this: If the original mapping is a function, will the reverse mapping necessarily be a function also? In other words, given a function, does there exist another function that precisely reverses the effect of the given function? We shall see that this is not always the case.

The definition of a function (Definition 1-1) requires that to each element of the domain there corresponds exactly one element of the range; it is perfectly all right for several elements of the domain to be mapped onto the same element of the range (the constant function, for example, maps all of its domain onto one element), but if even one element of the domain is mapped onto more than one element of the range, then the mapping just isn't a function. In terms of a picture of a function as a mapping (such as Figures 1-1a and 1-1c), this means that no two arrows may start from the same point, though any number of them may end at the same point. But if two or more arrows go to one point, as in Figure 1-6c, and if we then reverse the arrows, as in Figure 1-6d, we will have two or more arrows starting from that point (as in Figure 1-1b), and the resulting mapping is not a function. Since the word "inverse" is used to describe only a mapping which is a function, we can conclude that not every function has an inverse.

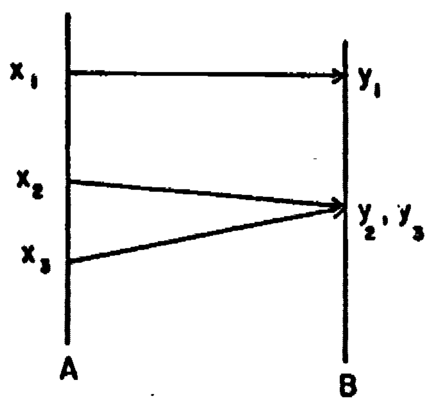


Figure 1-6c.

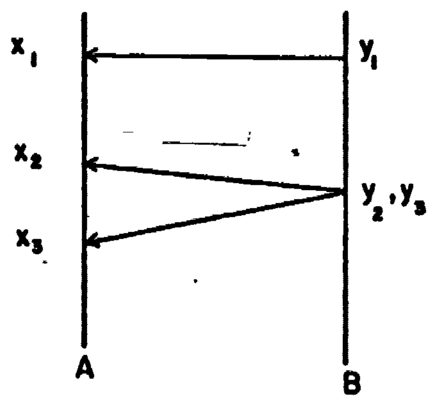


Figure 1-6d.

A specific example is furnished by the constant function $f: x \longrightarrow 3$; since $f(0) = 3$ and $f(1) = 3$, an inverse of f would have to map 3 onto both 0 and 1. By definition, no function can do this.

Activity # 1

For each problem draw a diagram that illustrates the function as in the example $f(x) = x$ in the text. For each problem the domain is the set of all real numbers. State by rule or roster the range of each function. Then graph each function.

1. $f(x) = x + 3$
2. $f(x) = 2x + 5$
3. $f(x) = |x| - 1$
4. $f(x) = |1 - x|$
5. $f(x) = x^2 + 1$

6. Why do you think $f(x) = 7$ is called a constant function?

In your textbook p. 210, # 25,27,29,31,33,35.

Activity # 2

1. Which of the following do not describe functions, when $x, y \in R$?
   a) $f: x \longrightarrow 3x - 4$
   b) $f: x \longrightarrow x^3$
   c) $f: x \longrightarrow -x$
   d) $f: x \longrightarrow y = x^2$
   e) $f: x \longrightarrow$ all $y < x$
   f) $f: x \longrightarrow 5x$
   g) $f: x \longrightarrow 16 - x^2$

2. Depict the mapping of a few elements of the domain into elements of the range for each of the Exercises 1(a), (c), and (d) above, as was done in Figure 1-1d.

3. Specify the domain and range of the following functions, where $x, f(x) \in R$.
   a) $f: x \longrightarrow x$
   b) $f: x \longrightarrow x^2$
   c) $f: x \longrightarrow \sqrt{x}$
   d) $f: x \longrightarrow \frac{x}{x - 1}$
   e) $f: x \longrightarrow \frac{3}{x^2 - 4}$

4. If $f: x \longrightarrow 2x + 1$, find
   a) $f(0)$
   b) $f(-1)$
   c) $f(100)$
   d) $f(\frac{3}{2})$

5. Given the function $f: x \longrightarrow x^2 - 2x + 3$, find
   a) $f(0)$
   b) $f(-1)$
   c) $f(a)$
   d) $f(x - 1)$

6. If $f(x) = \sqrt{x^2 - 16}$, find
   a) $f(4)$
   b) $f(-5)$
   c) $f(5)$
   d) $f(a)$
   e) $f(a - 1)$
   f) $f(\pi)$

7. If $x \in R$, given the functions

$$f: x \longrightarrow x$$

and

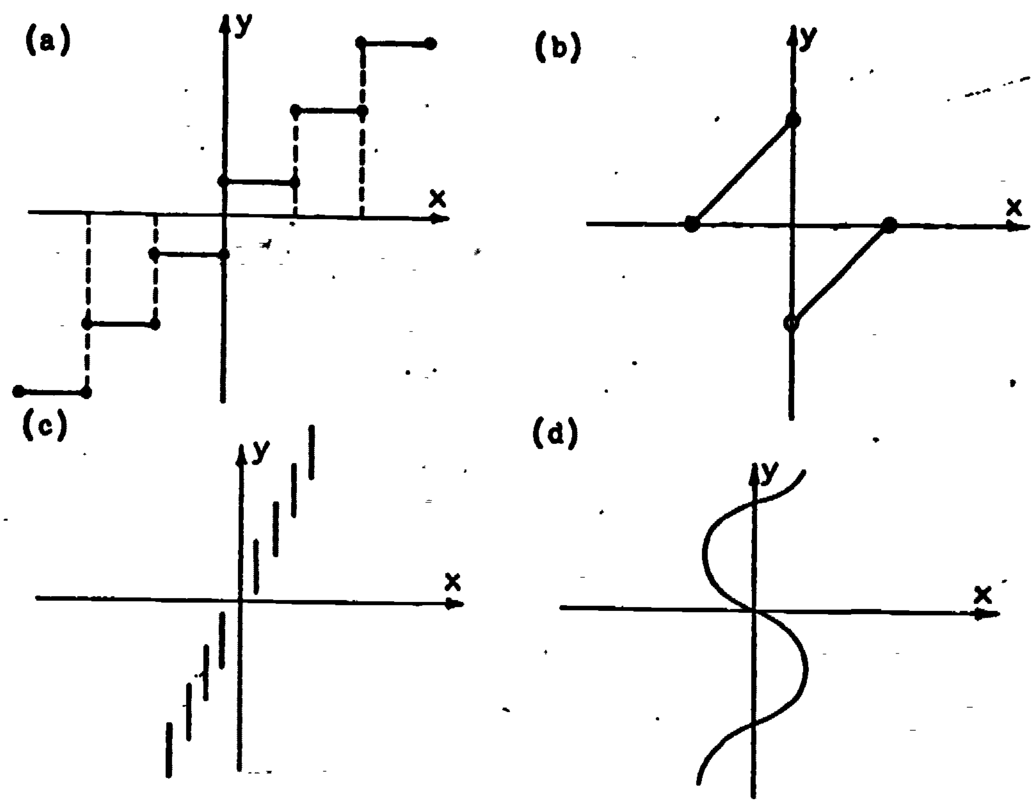
$$g: x \longrightarrow \frac{x^2}{x}$$

are f and g the same function? Why or why not?

8. What number or numbers have the image 16 under the following functions?
   a) $f: x \longrightarrow x^2$
   b) $f: x \longrightarrow 2x$
   c) $f: x \longrightarrow \sqrt{x^2 + 112}$

Activity # 3

1. Which of the following graphs could represent functions?



2. Suppose that in (a) above, f: $x \longrightarrow f(x)$ is the function whose graph is depicted. Sketch
   a) g: $x \longrightarrow -f(x)$ b) g: $x \longrightarrow f(-x)$
3. Graph the following functions.
   a) f: $x \longrightarrow 2x$
   b) f: $x \longrightarrow \frac{1}{x}$
   c) f: $x \longrightarrow y = 4 - x$ and x and y are positive integers.
   d) f: $x \longrightarrow -\sqrt{4 - x^2}$
4. Graph the following functions and indicate the domain and range of each by heavy lines on the x-axis and y-axis respectively.
   a) f: $x \longrightarrow y = x$ and $2 < y < 3$
   b) f: $x \longrightarrow \sqrt{9 - x^2}$
   c) f: $x \longrightarrow \sqrt{x}$ and $x < 4$

Activity # 4

1. Find the slope of the function f if, for all real numbers x,
   a) $f(x) = 3x - 7$
   b) $f(x) = 6 - 2x$
   c) $2f(x) = 3 - x$
   d) $3f(x) = 4x - 2$
2. Find a linear function f whose slope is -2 and such that
   a) $f(1) = 4$ c) $f(3) = 1$
   b) $f(0) = -7$ d) $f(8) = -3$
3. Find the slope of the linear function f if $f(1) = -3$ and
   a) $f(0) = 4$ c) $f(5) = 5$
   b) $f(2) = 3$ d) $f(6) = -13$
4. Find a function whose graph is the line joining the points
   a) $P(1, 1)$, $Q(2, 4)$ c) $P(1, 3)$, $Q(1, 8)$
   b) $P(-7, 4)$, $Q(-5, 0)$ d) $P(1, 4)$, $Q(-2, 4)$
5. Given $f: x \longrightarrow -3x + 4$, find a function whose graph is parallel to the graph of f and passes through the point
   a) $P(1, 4)$ c) $P(1, 5)$
   b) $P(-2, 3)$ d) $P(-3, -4)$
6. If f is a constant function find $f(3)$ if
   a) $f(1) = 5$
   b) $f(8) = -3$
   c) $f(0) = 4$
7. Do the points $P(1, 3)$, $Q(3, -1)$, and $S(7, -9)$ all lie on a single line? Prove your assertion.
8. The graph of a linear function f passes through the points $P(100, 25)$ and $Q(101, 39)$. Find
   a) $f(100.1)$
   b) $f(100.3)$
9. The graph of a linear function f passes through the points $P(53, 25)$ and $Q(54, -19)$. Find
   a) $f(53.3)$
   b) $f(53.8)$
10. Find a linear function with graph parallel to the line with equation $x - 3y + 4 = 0$ and passing through the point of intersection of the lines with equations $2x + 7y + 1 = 0$ and $x - 2y + 8 = 0$.
11. Given the points $A(1, 2)$, $B(5, 3)$, $C(7, 0)$, and $D(3, -1)$, prove that ABCD is a parallelogram.
12. Find the coordinates of the vertex C of the parallelogram ABCD if AC is a diagonal and the other vertices are the points:
   a) $A(1, -1)$, $B(3, 4)$, $D(2, 3)$;
   b) $A(0, 5)$, $B(1, -7)$, $D(4, 1)$

Activity # 5

1. a) For what $x \in R$ is it true that $\sqrt{x^2} = x$?
   b) For what $x \in R$ is it true that $\sqrt{x^2} = -x$?
2. a) For what $x \in R$ is it true that $|x - 1| = x - 1$?
   b) For what $x \in R$ is it true that $|x - 1| = -x + 1$?
   c) Sketch a graph of $f: x \longrightarrow |x - 1|$.
   d) Sketch a graph of $f: x \longrightarrow |x| - 1$.
3. Solve:
   a) $|x| = 14$
   b) $|x + 2| = 7$
   c) $|x - 3| = -1$
4. For what values of x is it true that
   a) $|x - 2| < 1$
   b) $|x - 5| > 2$
   c) $|x + 4| < 0.2$
   d) $|2x - 3| < 0.04$
   e) $|4x + 5| < 0.12$
5. Show that $x^2 \geq x \cdot |x|$ for all $x \in R$.
6. Show that $|a - b| \leq |a| + |b|$.
7. Show that $\frac{1}{2}(a + b + |a - b|)$ is equal to the greater of a and b. Can you write a similar expression for the lesser of a and b?
8. Sketch: $y = |x| + |x - 2|$. (Hint: you must consider, separately, the three possibilities $x < 0$, $0 \leq x < 2$, and $x \geq 2$.)

Activity # 6

1. Find an inverse of each of the following functions:
   a) $x \longrightarrow x - 7$
   b) $x \longrightarrow 5x + 9$
   c) $x \longrightarrow 1/x$
2. Solve each of the following equations for x in terms of y and compare your answers with those of Exercise 1:
   a) $y = x - 7$
   b) $y = 5x + 9$
   c) $y = 1/x$
3. Justify the following in terms of composite functions and inverse functions: Ask someone to choose a number, but not to tell you what it is. "Ask the person who has chosen the number to perform in succession the following operations. (i) To multiply the number by 5. (ii) To add 6 to the product. (iii) To multiply the sum by 4. (iv) To add 9 to the product. (v) To multiply the sum by 5. Ask to be told the result of the last operation. If from this product 165 is subtracted, and then the difference is divided by 100, the quotient will be the number thought of originally."
   (W. W. Rouse Ball).